Auxiliary Material for "Use of Breeding to Detect and Explain Instabilities in the Global Ocean"

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1 Bred Vector Energy Equation Derivations

Here we derive the bred vector kinetic energy equation and then the bred vector potential energy equation. Notationally, we write the velocity vector of the control run as $\vec{V}_c$. The velocity of the perturbed run is then $\vec{V}_c + \vec{V}_b$, where $\vec{V}_b$ is the bred vector velocity.

1.1 Bred Vector Kinetic Energy

The MOM2 horizontal momentum equations for the control run are as follows:

$$\begin{aligned}\frac{\partial u_c}{\partial t} &= -\vec{V}_c \cdot \bigtriangledown u_c - w_c u_{c_z} + f v_c + \frac{u_c v_c \tan\phi}{a} \\ &\quad - \frac{p_{c_\lambda}}{a\rho_0 \cos\phi} + (\kappa_m u_{c_z})_z + F^{u_c} \qquad (1)\\ \frac{\partial v_c}{\partial t} &= -\vec{V}_c \cdot \bigtriangledown v_c - w_c v_{c_z} - f u_c - \frac{u_c^2 \tan\phi}{a} \\ &\quad - \frac{p_{c_\phi}}{a\rho_0} + (\kappa_m v_{c_z})_z + F^{v_c} \qquad (2)\end{aligned}$$

This means that the momentum equations for the perturbed case are:

$$\begin{aligned}\frac{\partial(u_c+u_b)}{\partial t} &= -(\vec{V}_c + \vec{V}_b) \cdot \bigtriangledown (u_c+u_b) \\ &\quad -(w_c+w_b)(u_c+u_b)_z + f(v_c+v_b) \qquad (3)\\ &\quad + \frac{(u_c+u_b)(v_c+v_b)\tan\phi}{a} \qquad (4)\\ &\quad - \frac{(p_c+p_b)_\lambda}{a\rho_0\cos\phi} + [\kappa_m (u_c+u_b)_z]_z + F^{u_c+u_b} \qquad (5)\\ \frac{\partial(v_c+v_b)}{\partial t} &= -(\vec{V}_c + \vec{V}_b) \cdot \bigtriangledown (v_c+v_b) - (w_c+w_b)(v_c+v_b)_z - f(u_c+u_b) \\ &\quad - \frac{(u_c+u_b)^2\tan\phi}{a} - \frac{(p_c+p_b)_\phi}{a\rho_0} + [\kappa_m (v_c+v_b)_z]_z + F^{v_c+v_b} \qquad (6)\end{aligned}$$

The bred vector momentum equations are then the difference between the perturbed momentum equations and the control momentum equations. Taking this difference yields:

$$\begin{aligned}\frac{\partial u_b}{\partial t} &= -\vec{V}_c\cdot\bigtriangledown u_b - \vec{V}_b\cdot\bigtriangledown u_c - \vec{V}_b\cdot\bigtriangledown u_b - w_c u_{b_z} - w_b u_{c_z} - w_b u_{b_z} \\ &\quad + f v_b + \frac{(u_c v_b + u_b v_c + u_b v_b)\tan\phi}{a} - \frac{p_{b_\lambda}}{a\rho_0\cos\phi} + (\kappa_m u_{b_z})_z + F^{u_b} \qquad (7)\\ \frac{\partial v_b}{\partial t} &= -\vec{V}_c\cdot\bigtriangledown v_b - \vec{V}_b\cdot\bigtriangledown v_c - \vec{V}_b\cdot\bigtriangledown v_b - w_c v_{b_z} - w_b v_{c_z} - w_b v_{b_z} \\ &\quad - f u_b + \frac{(2u_c u_b + u_b^2)\tan\phi}{a} - \frac{p_{b_\phi}}{a\rho_0} + (\kappa_m v_{b_z})_z + F^{v_b} \qquad (8)\end{aligned}$$

We define the bred vector kinetic energy, KE_b, as $KE_b = \frac{\rho_0}{2}\vec{V}_b\cdot\vec{V}_b$. Taking the dot product of $\vec{V}_b$ and $\frac{\partial\vec{V}_b}{\partial t}$ leads to the kinetic energy equation for the bred perturbation:

$$\begin{aligned}\frac{\partial KE_b}{\partial t} &= -\vec{V}_c\cdot\bigtriangledown KE_b - \rho_0\vec{V}_b\cdot(\vec{V}_b\cdot\bigtriangledown)\vec{V}_c - \vec{V}_b\cdot\bigtriangledown KE_b - w_c KE_{b_z} - \rho_0\vec{V}_b\cdot(w_b\vec{V}_{c_z}) - w_b KE_{b_z} - \frac{u_b p_{b_\lambda}}{a\cos\phi} \\ &\quad - \frac{v_b p_{b_\phi}}{a} + \frac{\rho_0(u_b^2 v_c - v_b u_c u_b)\tan\phi}{a} + \rho_0 u_b(\kappa_m u_b)_z + \rho_0 v_b(\kappa_m v_b)_z + \rho_0\vec{V}_b\cdot F^{\vec{V}_b} \qquad (9)\end{aligned}$$

Now we can rewrite this equation making a few approximations. First, a few of these terms are very small and can be ignored. The coriolis-related term is negligible ($\frac{\rho_0(u_b^2 v_c - v_b u_c u_b)\tan\phi}{a}\approx 0$) so we can disregard it. The term $w_b KE_{b_z}$ is, on average, two orders of magnitude smaller than the similar term $w_c KE_{b_z}$, while $\vec{V}_b\cdot\bigtriangledown KE_b$ is two to three orders of magnitude smaller than $\vec{V}_c\cdot\bigtriangledown KE_b$. Since $w_b \ll w_c$ and $\vec{V}_b \ll \vec{V}_c$, the terms $w_b KE_{b_z}$ and $\vec{V}_b\cdot\bigtriangledown KE_b$ can be ignored. Next, we write $-\frac{u_b p_{b_\lambda}}{a\cos\phi} - \frac{v_b p_{b_\phi}}{a} = -\vec{V}_b\cdot\bigtriangledown p_b$ and $\rho_0 u_b(\kappa_m u_b)_z + \rho_0 v_b(\kappa_m v_b)_z + \rho_0\vec{V}_b\cdot F^{\vec{V}_b} = \vec{F}_b$. Equation 9 then becomes:

$$\frac{\partial KE_b}{\partial t} = -\vec{V}_c\cdot\bigtriangledown KE_b - \rho_0\vec{V}_b\cdot(\vec{V}_b\cdot\bigtriangledown)\vec{V}_c - w_c e_{b_z} - \rho_0\vec{V}_b\cdot(w_b\frac{\partial\vec{V}_c}{\partial z}) + \vec{V}_b\cdot\bigtriangledown p_b + \vec{F}_b \qquad (10)$$

Next we expand some of these terms. First,

$$-\vec{V}_c\cdot\bigtriangledown KE_b = -\bigtriangledown\cdot(\vec{V}_c KE_b) + KE_b\bigtriangledown\cdot\vec{V}_c = -\bigtriangledown\cdot(\vec{V}_c KE_b) - KE_b w_{c_z}.$$

The last equality comes from the continuity equation, $\bigtriangledown\cdot\vec{V} + w_z = 0$. Also, using the continuity equation and the hydrostatic approximation,

$$\begin{aligned}-\vec{V}_b\cdot\bigtriangledown p_b &= -\bigtriangledown\cdot(\vec{V}_b p_b) + p_b\cdot\bigtriangledown\vec{V}_b &&= -\bigtriangledown\cdot(\vec{V}_b p_b) - p_b w_{b_z} \\ &= -\bigtriangledown\cdot(\vec{V}_b p_b) - \tfrac{\partial(w_b p_b)}{\partial z} + w_b\tfrac{\partial p_b}{\partial z} &&= -\bigtriangledown\cdot(\vec{V}_b p_b) - \tfrac{\partial(w_b p_b)}{\partial z} - w_b g\rho_b.\end{aligned}$$

Making these two substitutions and grouping the terms yields the following bred vector kinetic energy equation:

$$\begin{aligned}\frac{\partial KE_b}{\partial t} &= -[\bigtriangledown\cdot(\vec{V}_c KE_b) + \frac{\partial}{\partial z}(w_c KE_b)] - [\bigtriangledown\cdot(\vec{V}_b p_b) + \frac{\partial}{\partial z}w_b p_b] \\ &\quad - w_b g\rho_b - \rho_0[\vec{V}_b\cdot(\vec{V}_b\cdot\bigtriangledown)\vec{V}_c + \vec{V}_b\cdot(w_b\frac{\partial\vec{V}_c}{\partial z})] + \vec{F}_b \qquad (11)\end{aligned}$$

The two Barotropic Energy Conversion terms (the fourth term in square brackets) can be written as the single term $-\vec{V}_b\cdot(\vec{V}_{b_3}\cdot\bigtriangledown_3)\vec{V}_c$, where $\bigtriangledown_3 = (\bigtriangledown, \frac{\partial}{\partial z})$ and $\vec{V}_{b_3} = (\vec{V}_b, w_b)$.

1.2 Bred Vector Potential Energy

The bred vector potential energy equation is derived in an analogous manner to the bred vector kinetic energy equation. Here we begin with the mass conservation equation

$$\frac{D\rho}{Dt} + \rho\bigtriangledown_3\cdot\vec{V} = 0 \qquad (12)$$

Expanding equation 12 and using the continuity equation yields:

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0 \tag{13}$$

For this calculation we break the density into a perturbation density, ρ', from an equilibrium solution, $\rho_0(z)$, so $\rho = \rho_0 + \rho'$. Plugging this into equation 13 gives us a lot of terms, but we can neglect half of them because of the scale. For the time and horizontal derivatives, the derivative of the perturbation density is much larger than the derivative of the equilibrium density, so we can ignore the terms involving ρ_{0_t}, ρ_{0_x}, and ρ_{0_y}. On the other hand, the derivative of the equilibrium density is the dominant term in the vertical direction, so we ignore the term involving ρ'_z. In addition, the density is split into control and bred vector pieces. We assume ρ_0 is the same for both runs, so the only difference is in the perturbation density. Thus instead using ρ', we write $\rho_{control} = \rho_0 + \rho_c$ and $\rho_{perturbed} = \rho_0 + \rho_c + \rho_b$. This leaves the following equation for the control run:

$$\rho_{c_t} + u_c\rho_{c_x} + v_c\rho_{c_y} + w_c\rho_{0_z} = 0 \tag{14}$$

For the perturbed run, the velocities are again the sum of the control and bred vector values. After making these assumption, the equation for the perturbed run becomes:

$$\rho_{c_t} + \rho_{b_t} + (u_c + u_b)(\rho_{c_x} + \rho_{b_x}) + (v_c + v_b)(\rho_{c_y} + \rho_{b_y}) + (w_c + w_b)\rho_{0_z} = 0 \tag{15}$$

Subtracting the control run equation from the perturbed run equation gives us the bred vector equation:

$$\rho_{b_t} + u_c\rho_{b_x} + u_b\rho_{c_x} + u_b\rho_{b_x} + v_c\rho_{b_y} + v_b\rho_{c_y} + v_b\rho_{b_y} + w_b\rho_{0_z} = 0 \tag{16}$$

To calculate the potential energy change, we multiply through by $\frac{\rho_b g^2}{\rho_0 N^2}$. Letting the bred vector potential energy be $PE_b = \frac{\rho^2 g^2}{2\rho_0 N^2}$ leads to the following bred vector potential energy equation:

$$\frac{\partial PE_b}{\partial t} = -\vec{V}_c \cdot \triangledown PE_b - \vec{V}_b \cdot \triangledown PE_b - \frac{g^2\rho_b}{\rho_0 N^2}\vec{V}_b \cdot \triangledown\rho_c - \frac{g^2 w_b\rho_b}{\rho_0 N^2}\rho_{0_z} \tag{17}$$

The first and second terms in this equation are both advection of kinetic energy. Since $\vec{V}_c \gg \vec{V}_b$, we can ignore the advection by the bred vector velocity and just keep the first term. As we did in deriving the kinetic energy equation, we rewrite the first term as

$$-\vec{V}_c \cdot \triangledown PE_b = -\triangledown \cdot (\vec{V}_c PE_b) + PE_b \triangledown \cdot \vec{V}_c = -\triangledown \cdot (\vec{V}_c PE_b) + PE_b w_{c_z} \tag{18}$$

For the third term, we can rewrite

$$-\vec{V}_b \cdot \triangledown\rho_c = -\triangledown \cdot (\vec{V}_b\rho_c) + \rho_c \cdot \triangledown\vec{V}_b = -\triangledown \cdot (\vec{V}_b\rho_c) - \rho_c w_{b_z} = -\triangledown \cdot (\vec{V}_b\rho_c) - \frac{\partial(w_b\rho_c)}{\partial z} + w_b\rho_{c_z}. \tag{19}$$

As discussed above, the vertical derivative of ρ_c is negligible as compared to the vertical derivative of ρ_0, so the $w_b\rho_{c_z}$ piece of the above equation can be ignored. Finally, N^2 is defined as $N^2 = -\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$, so $\frac{\partial\rho_0}{\partial z} = -\frac{\rho_0 N^2}{g}$. Substituting this into the fourth term yields

$$-\frac{g^2 w_b\rho_b}{\rho_0 N^2}\rho_{0_z} = \frac{g^2 w_b\rho_b}{\rho_0 N^2}\frac{\rho_0 N^2}{g} = w_b g\rho_b. \tag{20}$$

After rewriting all of the terms, the final bred vector potential energy equation is

$$\frac{\partial PE_b}{\partial t} = -[\triangledown \cdot (\vec{V}_c PE_b) + \frac{\partial}{\partial z}(w_c PE_b)] + w_b g\rho_b - \frac{g^2\rho_b}{\rho_0 N^2}[\triangledown \cdot (\vec{V}_b\rho_c) + \frac{\partial}{\partial z}(w_b\rho_c)] - w_c\frac{\partial PE_b}{\partial z} \tag{21}$$

As was already mentioned, the first term is advection of potential energy by the control velocity. The second term is the conversion of energy from potential to kinetic. It is the same term that showed up in equation 11 only with opposite sign.