Local Low Dimensionality of Atmospheric Dynamics

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A statistic, the *BV-dimension*, is introduced to measure the effective local finite-time dimensionality of a spatio-temporally chaotic system. It is shown that the Earth's atmosphere often has low BV-dimension [1,2]. Furthermore, as we discuss at the end of this paper, we believe that this finding has important implications for weather forecasting.

On December 7, 1992 the NWS implemented operational *ensemble forecasts* [3]. At regular time intervals several perturbations to the model atmospheric state are created. The original atmospheric state (referred to below as the *main solution*) and the ensemble of perturbed states are evolved forward in time by the model to create an ensemble of forecasts.

The difference between the main solution and a perturbed solution is similar to a Lyapunov vector in the calculation of the evolution of differential displacements from chaotic trajectories [4,5]. A difference here is that for the NWS computation, the perturbations, although small, are not infinitesimal. The individual perturbations obtained from the ensemble forecasts are called the *bred vectors*. If the perturbations were infinitesimal (rather than finite), then, in the limit of infinite time evolution, the bred vectors would point in the direction of dominant growth, and the corresponding exponential growth rate of their magnitudes would be the largest Lyapunov exponent.

For the analysis presented in this paper, we used ensembles consisting of 5 perturbed forecasts [5,6]. The ensemble forecasts are made available on the Internet every 24 hours by the NWS and give the forecasts at 12 hour intervals spanning 8 days. In this study we focus on the wind vector field at the height where the pressure is 500 millibars (approximately 5 km in altitude).

We consider square regions of roughly 1100 km x 1100 km, choosing a grid point in the center of the region plus 24 uniformly distributed neighbors. The north-south and east-west wind components of a bred vector at the 500mb pressure level at each of the 25 points in such a region form a 50 dimensional column vector which we refer to as a *local bred vector*.

If there are k local bred vectors ($k = 5$ in our case), the issue we want to address is the degree of linear independence of these k local bred vectors. That is, we want to determine the effective dimensionality of the subspace spanned by the local bred vectors. To do this we use empirical orthogonal functions (EOFs) (also known as principal component analysis). The underlying idea is to find the lowest dimensional subspace that, in a least squares sense, optimally represents the majority of the data.

The k local bred column vectors form a 50xk matrix, B. The singular values of B are σ_i and measure the extent to which the k column unit vectors making up B point in the direction v_i. By identifying how many of the vectors of v_i represent most of the variance of B we can identify an effective dimension spanned by the k local bred vectors. In order to do so, we define the following statistic on the singular values which we call the *BV-dimension* (bred vector dimension):

$$\psi(\sigma_1, \sigma_2, \ldots, \sigma_k) = \frac{(\sum_{i=1}^{k} \sigma_i)^2}{\sum_{i=1}^{k} \sigma_i^2}. \qquad (1)$$

As examples of the statistic, we describe several cases with $k = 5$ unit vectors. If the k local bred vectors comprising B were all the same, then the singular values would be $\sqrt{5}, 0, 0, 0, 0$. This would yield a statistic of $\psi(\sqrt{5}, 0, 0, 0, 0) = 1$. If the local bred vectors were equally distributed between two vectors v_1 and v_2, in the sense that each one accounts for half the variance, then the singular values would be $\sqrt{5/2}, \sqrt{5/2}, 0, 0, 0$, and our statistic would yield $\psi(\sqrt{5/2}, \sqrt{5/2}, 0, 0, 0) = 2$. If the local bred vectors again lie in the two dimensional subspace spanned by v_1 and v_2, but the two are not equally represented, then this could give $2 \geq \psi \geq 1$ [e.g., we might have singular values such as $\sqrt{4.75}, 1/2, 0, 0, 0$ which would yield $\psi(\sqrt{4.75}, 1/2, 0, 0, 0) \approx 1.4$]. While the dimension of the space spanned by the local bred vectors is 2, our statistic gives an intermediate value reflecting the degree of dominance of one direction over the other. In general our statistic returns a real value between 1 and k.

In Figure 1 we show this analysis applied to a 36 hour forecasts. The BV-dimension was calculated at each spatial point on the grid and colored red for lower values and blue for larger values. A large region of relatively low dimensionality (BV-dimension less than 2.5) is evident over North America (red-yellow). This indicates that in this region, the local bred vectors effectively span a space of substantially lower dimension than that of the full space. We also find that there is a well defined vertical structure (from 850 millibars to 250 millibars) in the atmospheric column of regions with low BV-dimensions. It is also found that low values of BV-dimension, over time, tend to follow fairly defined tracks in our data set moving from west to east. The average location of regions with low BV-dimension for the time period of February 10, 2000 to July 30, 2000 is exhibited in Figure 2 [7].

The regions of low BV-dimension have potentially important implications for weather forecasting. A major effort in forecasting is devoted to the process of data assimilation. At any given time t_0, there is inevitably a discrepancy $\vec{\Delta}(t_0)$ between the true atmospheric state and its representation in the computer model. Now consider a later time $t_1 > t_0$, and suppose that in a region of interest there is a low BV-dimension at time t_1. This implies that any local discrepancy $\vec{\Delta}(t_1)$ between the true state and its representation in the computer model lies predominantly in the "unstable subspace", the space spanned by the few vectors that contribute most strongly to the the low BV-dimension [8]. We conjecture that in many cases this information can yield a substantial improvement in forecasting. In particular, the implication is that the data assimilation algorithm should correct the computer model state by moving it closer to the observations along the direction of the unstable subspace since that is where the true state most likely lies [9]. Current data assimilation techniques do not take this into account.

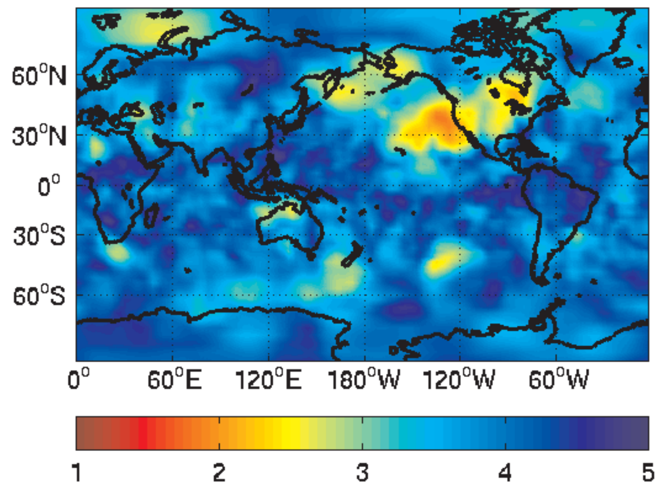

FIG. 1. An example of the spatial variation of BV-dimension from a 36 hour forecast from March 5, 2000. The colors represent the effective subspace spanned by the local bred vectors (BV-dimension), with red indicating lower values and blue indicating higher values.

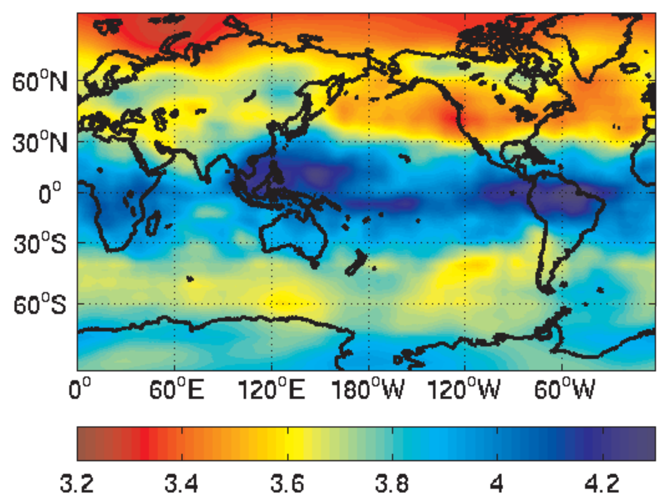

FIG. 2. Average locations of regions with low BV-dimensions are shown through the point-wise time average of the BV-dimension calculated from ensemble forecasts every 12 hours from February 10, 2000 to July 30, 2000. Red (blue) depicts regions in which the BV-dimension tends to be low (high).

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[1] D.J. Patil, B.R. Hunt, E. Kalnay, E. Ott, J. Yorke, Local Low Dimensionality of Atmospheric Dynamics, To Appear in Phys. Rev. Letters.

[2] The type of low dimensionality that we consider is very different from the low dimensionality that has been reported in the past from the application of embedding theorems to time series data. Such results have been shown to be in doubt due to the misinterpretation of results.

[3] Z. Toth and E. Kalnay, Bulletin of the American Meteorological Society **72,** 2317 (1993).

[4] K.T. Alligood, T.D. Sauer, and J.A. Yorke, *Chaos: An Introduction to Dynamical Systems* (Springer , New York, 1997); E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993).

[5] Z. Toth and E. Kalnay, Monthly Weather Review **125,** 3297 (1997).

[6] 10 bred vectors are made available, but 5 of them are the result of subtracting the perturbations from the main solution, hence we average the pairs of the perturbations to give 5 bred vectors.

[7] The locations of regions with low BV-dimension show a striking resemblance patterns of low dimensional atmospheric instabilities identified by D.J. Patil, B.R. Hunt, and J.A. Carton, 1999, To be published.

[8] D.J. Patil, B.R. Hunt, E. Kalnay, E. Ott, J. Yorke, Evaluating the Independence of Bred Vectors in Ensemble Forecasts Over Geographically Localized Regions, Preprint.

[9] E. Kalnay and Z. Toth, Preprints; Tenth Conference on Numerical Weather Prediction (July 18-22, 1994)