6.2 Brief review of fundamental concepts about chaotic systems

Lorenz (1963) introduced a 3-variable model that is a prototypical example of chaos theory. These equations were derived as a simplification of Saltzman's (1962) nonperiodic model for convection. Like Lorenz's (1962) original 12-variable model, the 3-variable model is a **dissipative** system. This is in contrast to **Hamiltonian** systems, which **conserve total energy** or some other similar property of the flow. The system is **nonlinear** (it contains products of the dependent variables) but **autonomous** (the coefficients are time-independent). Sparrow (1982) wrote a whole book on the Lorenz 3-variable model that provides a nice introduction to the subject of chaos, bifurcations and strange attractors. Lorenz (1993) is a superbly clear introduction to chaos with a very useful glossary of the nomenclature used in today's literature. Alligood et al (1996) is also a very clear introduction to dynamical systems and chaos. In this section we use bold type to introduce some of the words used in the dynamical system vocabulary.

The Lorenz (1963) equations are

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz\end{aligned} \tag{0.1}$$

The solution obtained by integrating the differential equations in time is called a **flow**. The **parameters** σ, b, r are kept constant within an integration, but they can be changed to create a **family of solutions** of the **dynamical system** defined by the differential equations. The particular parameter values chosen by Lorenz (1963) were $\sigma = 10,\ b = 8/3,\ r = 28$, which result in **chaotic** solutions (sensitively dependent on the initial conditions), and since this publication they have been widely used in many papers. The solution of a time integration from a given **initial condition** defines a **trajectory** or **orbit** in **phase space**. The coordinates of a **point in phase space** are defined by the simultaneous values of the independent variables of the model, *x(t), y(t), z(t)*. The **dimension of the phase space** is equal to the number of independent variables (in this case 3). The dimension of the subspace actually visited by the solution after an initial transient period (i.e., the **dimension of the attractor**) can be much smaller than the dimension of the phase space. A **volume** in phase space can be defined by a set of points in phase space such as a hypercube $V = \delta x \delta y \delta z$, a hypersphere $V = \{\delta \mathbf{r}; |\delta \mathbf{r}| \le \varepsilon\}$, etc.

The fact that the Lorenz system (0.1) is dissipative can be seen from the **divergence** of the **flow:**

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + b + 1) \qquad (0.2)$$

which shows that an original volume *V* contracts with time to $Ve^{-(\sigma+b+1)t}$. This proves the existence of a **bounded**

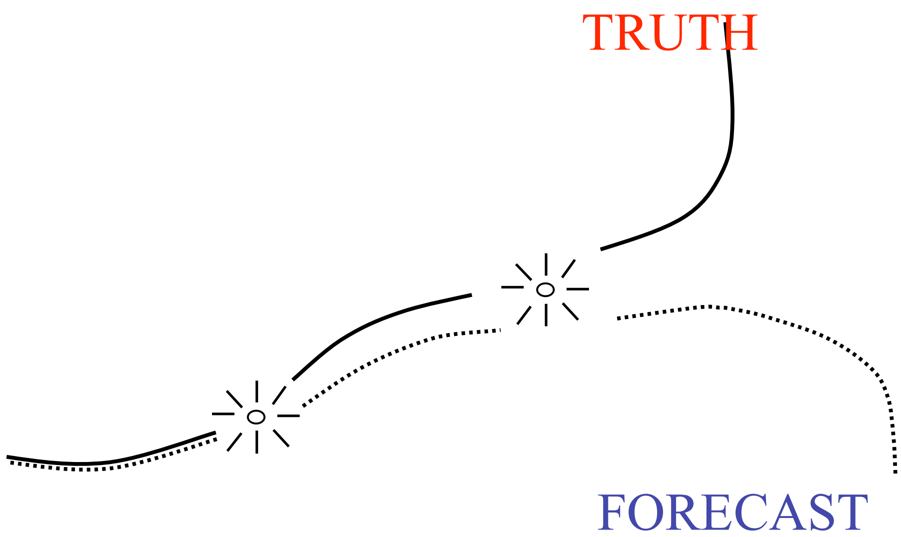
globally attracting set of zero volume (i.e., an attractor of **dimension** smaller than *n*, the dimension of the phase space). A solution may start from a point away from the attracting set but it will eventually settle on the **attractor**. This initial portion of the trajectory is known as a **transient**. The **attracting set** (the set of points approached again and again by the trajectories after the transients are over) is called the **attractor** of the system. The attractor can have several components: **stationary** points (equilibrium or steady state solutions of the dynamical equations), **periodic orbits**, and more complicated structures known as **strange attractors** (which can also include periodic orbits). The different components of the attractor have corresponding **basins of attraction** in the phase space, which are all the initial conditions that will evolve to the same attractor. The fact that any initial volume in phase space contracts to zero with time is a general property of dissipative bounded systems, including atmospheric models with friction. Hamiltonian systems, on the other hand, are **volume conserving**.

If we change the **parameters** of a dynamical system (in this example σ, b, r) and obtain families of solutions, we find that there is a point at which the behavior of the flow changes abruptly. The point at which this sudden change in the characteristics of the flow occurs is called a **bifurcation point**. For example in Lorenz' equations the origin is a stable, stationary point for $r<1$ as can be seen by investigating the local stability at the origin. The **local stability** of a point can be investigated by linearizing the flow about the point and computing the eigenvalues of the linear flow. For $r<1$ the stationary point is stable: all three eigenvalues are negative. This means that all orbits nearby the origin tend to get closer to it. At $r=1$ there is a bifurcation, and for *r>1* two new additional stationary points $C_{+/-}$ are born, with coordinates $(x,y,z)_{\pm} = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. For *r>1* the origin becomes **non-stable:** one of the three eigenvalues becomes positive (while the other two remain negative), indicating that the flow diverges locally from the origin in one direction. For 1<r<24.74…, C_+ and C_- are stable, and at r=24.74… there is another bifurcation so that above that critical value C_+ and C_- also become unstable. As discussed by Lorenz (1993), an ubiquitous phenomenon is the occurrence of bifurcations of periodic motion leading to **period doubling**, and **sequences of period doubling bifurcations** leading to chaotic behavior (see Sauer et al, 1991).

A solution of a dynamical system can be defined to be **stable** if it is bounded, and if any other solution once sufficiently close to it remains close to it for all times. This indicates that a bounded stable solution must be **periodic** (repeat itself exactly) or at least **almost periodic**, since once the trajectory approaches a point in its past history, the trajectories will remain close forever (Fig. 6.1b). A solution that is not periodic or almost periodic is therefore **unstable**: two trajectories that start very close will eventually diverge completely (Fig.6.1a).

The long-term stability of a dynamical system of n-variables is characterized by the **Lyapunov exponents**. Consider a point in a trajectory, and introduce a (hyper) sphere of small perturbations about that point. If we apply the model to evolve each of those perturbations, we find that after a short time the sphere will be deformed into a (hyper) ellipsoid. In an unstable system, at least one of the axis of the ellipsoid will become larger with time, and once nonlinear effects start to be significant the ellipsoid will be deformed into a "banana" (Fig. 6.2). Consider the linear phase, during which the sphere evolves into an ellipsoid. We can maintain the linear phase for an infinitely long period by taking an infinitely small initial sphere, or, alternatively, by periodically scaling down the ellipsoid dimensions dividing all its dimensions by the same scalar. Each axis *j* of the ellipsoid grows or decays over the long term by amounts given by $e^{\lambda_j t}$, where the λ_j 's are the Lyapunov exponents ordered by size $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The total volume of the ellipsoid will evolve like $V_0 e^{-(\lambda_1 + \lambda_2 ... \lambda_n)t}$. Therefore, a Hamiltonian (volume-conserving) system is characterized by a sum of Lyapunov exponents equal to zero, whereas for a dissipative system the sum is negative.

a) Unstable dynamical system



b) Stable dynamical system

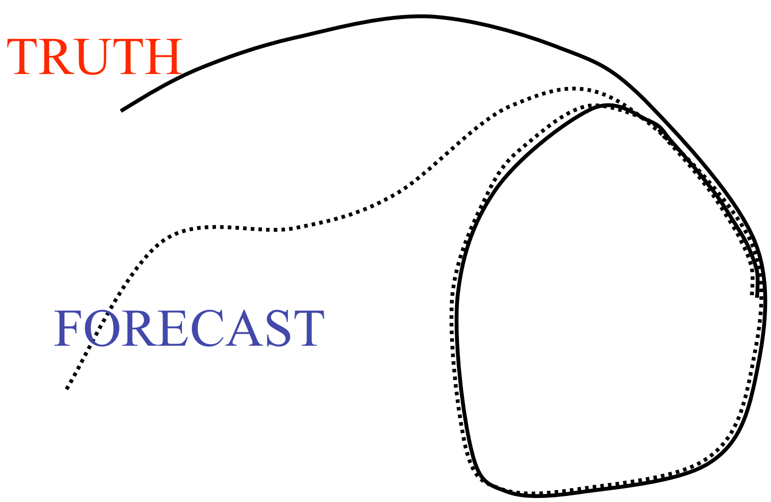
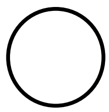
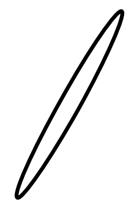


Fig. 6.2: Schematic of the evolution of a small spherical volume in phase space in a bounded dissipative system. Initially (during the linear phase) the volume gets stretched into an ellipsoid while the volume decreases. The solution space is bounded, and a bound is schematically indicated in the figure by the hypercube. The ellipsoid continues to be stretched in the unstable directions, until (because the solution phase space is bounded) it has to fold through nonlinear effects. This stretching and folding continues again and again, evolving into an infinitely foliated (fractal) structure. This structure, of zero volume and fractal dimension is called a "strange attractor". The attractor is the set of states whose vicinity the system will visit again and again (the “climate” of the system). Note that in phases a, b, and c, there is predictive knowledge: we know where the original perturbations generally are. In d, when the original sphere has evolved into the attractor, all predictability is lost: we only know that each original perturbation is within the climatology of possible solutions, but we don’t know where, or even in which region of the attractor it may be.

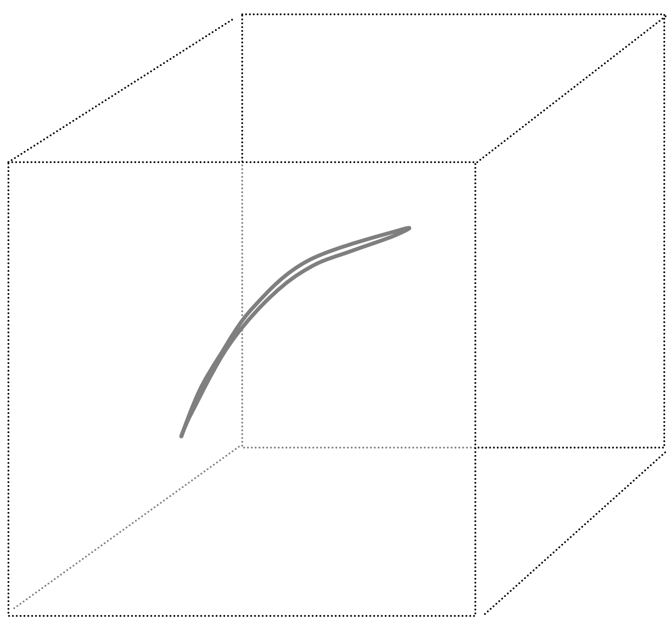
a) Initial volume: a small hypersphere



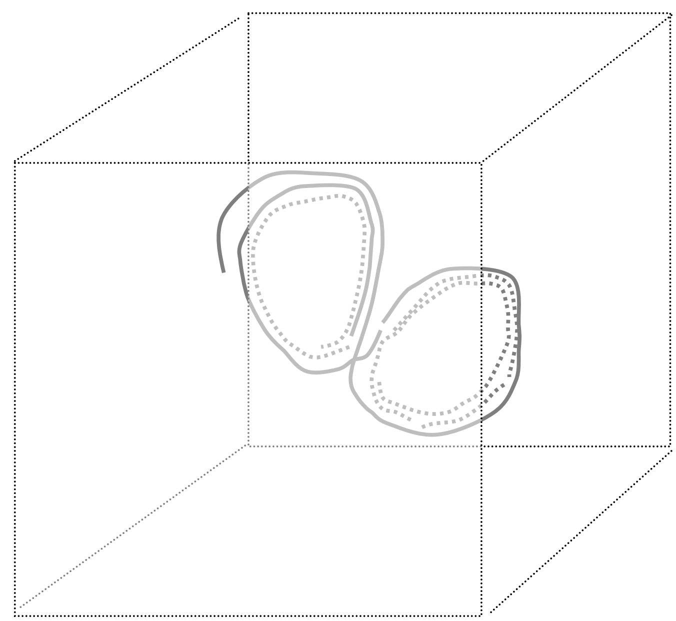
b) Linear phase: a hyper ellipsoid



c) Nonlinear phase: folding needs to take place in order for the solution to stay within the bounds



d) Asymptotic evolution to a strange attractor of zero volume and fractal structure. All predictability is lost



Because the attractor of a dissipative system is bounded (the trajectories are enclosed within some hyperbox), if the first Lyapunov is greater than zero, at least one of the axes of the ellipsoid keeps getting longer with time. The ellipsoid will eventually be distorted into a banana shape: it has to be folded in order to continue fitting into the box. The banana will be further stretched along the unstable axis and then necessarily folded again and again onto itself in order to continue fitting into the box. Since the volume of the ellipsoid eventually goes to zero for a dissipative system, the repeated stretching and folding of the ellipsoid of a chaotic system eventually converges to a zero-volume attractor with an infinitely foliated structure (a process similar to the stretching and folding used to make “phyllo” dough!). This structure is known as "**strange attractor**" (Ruelle, 1989). It has a **fractal** structure: a dimension which in general is not an integer and is smaller than the original space dimension *n*, estimated by Kaplan and Yorke (1979) to be

$$d = k + (\lambda_1 + ... + \lambda_k) / | \lambda_{k+1} | \tag{2.1}$$

where the sum of the first *k* Lyapunov exponents is positive, and the sum of the first *k+1* exponents is negative. If the system is Hamiltonian, its invariant manifold has the same dimension as the phase space.

In summary, a **stable** system has all Lyapunov exponents less or equal to zero. A **chaotic** system has **at least one Lyapunov exponent greater than zero:** if at least $\lambda_1 > 0$ chaotic behavior will take place because at least one axis of the ellipsoid will be continuously stretched, leading to the separation of orbits originally started closely along that axis. Note that a **chaotic bounded flow** must also have **a Lyapunov exponent equal to zero**, with the corresponding local Lyapunov vector parallel to an orbit. This can be understood by considering two initial conditions such that the second is equal to the first after applying the model for one time step. The solutions corresponding to these initial conditions will remain close together, since the second orbit will always be the same as the first orbit shifted by one time step, and on the average, the distance between the solutions will remain constant. If we add a tiny perturbation, though, the second solution will diverge from the first one because there is a positive Lyapunov exponent.

Fig. 6.7: Schematic of how all perturbations will converge towards the leading Local Lyapunov Vector