3.3.3: Semi-Lagrangian schemes

AOSC614 class

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Truly Lagrangian scheme:

$$\frac{du}{dt} = S(u)$$

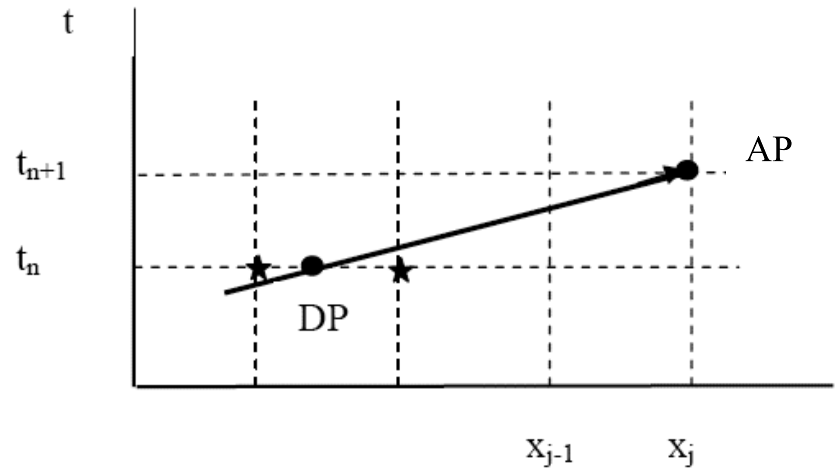
Following an individual parcel, the total derivative (also known as individual, substantial or Lagrangian time derivative) is conserved for a parcel, except for the changes introduced by the source or sink S.

- No grid points, No spatial discretization, no stability problem.
- However, It is not practical in general because one has to keep track of many individual parcels, and with time they may "bunch up" in certain areas of the fluid, and leave others without parcels to track.

The semi-Lagrangian scheme keeps the advantage of stability and avoids the disadvantage of parcels bunching up

Semi-Lagrangian scheme(1):

$$\frac{du}{dt} = S(u)$$

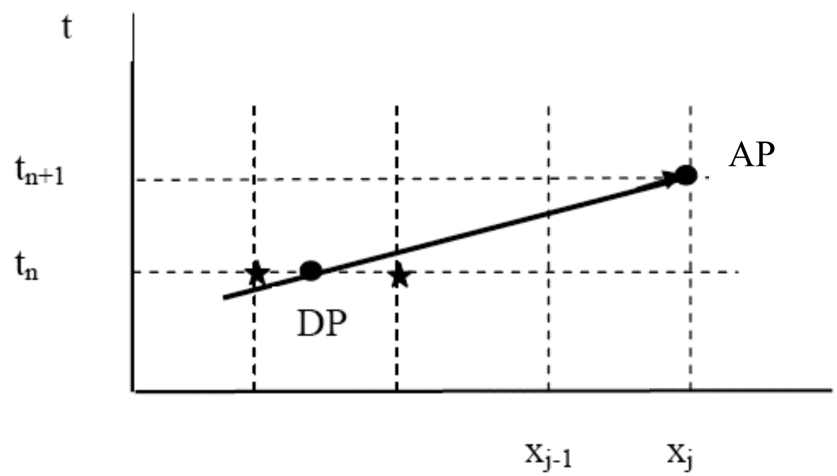


1) Using regular grid as in Eulerian coordinate, but estimate the total derivative.

2) At every new time step we find out where the parcel arriving at a grid point (denoted arrival point or AP) *came from* in the previous time step (denoted departure point or DP).

Semi-Lagrangian scheme:

$$\frac{du}{dt} = S(u)$$



- In a two-time level scheme it could be written as

$$(U_j^{n+1})_{AP} = (U^n)_{DP} + \frac{\Delta t}{2}[S(U^n)_{DP} + S(U_j^{n+1})_{AP}]$$

- Compare with Eulerian scheme: $$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + S(u)$$

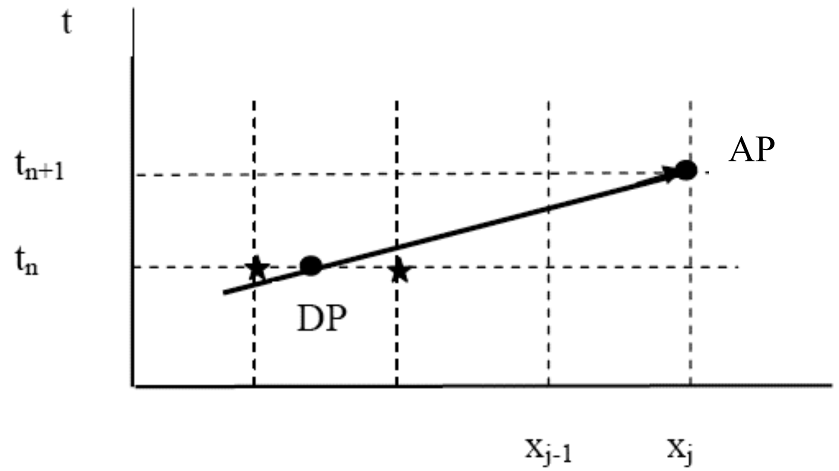
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = U_j^n(\delta_x U_j^n) + S(U_j^n)$$

- No extrapolation, the semi-Lagrangian scheme is absolutely stable with respect to advection.

- However, how do we know the location of the departure point DP ?

Semi-Lagrangian scheme(3):

$$\frac{du}{dt} = S(u)$$



The departure point DP has to be determined from the trajectory integrated between the departure and the arrival points, for example as

$$x_{DP} = x_{AP} - \frac{\Delta t}{2}(U_{DP} + U_{AP})$$

However, U_{AP} and U_{DP} are not known until the departure point has been determined, Therefore, it is an implicit equation that needs to be solved iteratively.

The accuracy of the SL scheme depends on the accuracy of the determination of the DP, and on the determination of the value of U_{DP} and the other conserved quantities by interpolation from the neighboring points.

3.3.4 Nonlinear computational instability. Quadratically conservative schemes. The Arakawa Jacobian

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Phillips (1957) quasi-geostrophic 2-level channel model

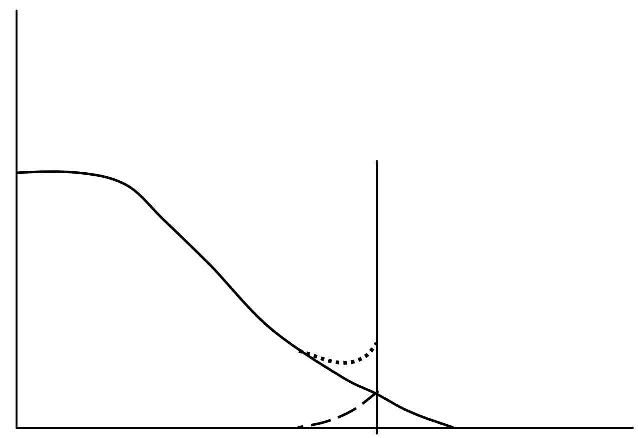
- In 1957 Phillips published the first "climate" or "general circulation" simulation ever made with a numerical model of the atmosphere.

- He obtained very realistic solutions that contributed significantly to the understanding of the atmospheric circulation in mid-latitudes.

- However, his climate simulation only lasted for about 16 days: the model "blew up" despite the fact that care had been taken to satisfy the von Neumann criterion for linear computational instability.

In 1959, Phillips pointed out that this instability, which he named nonlinear computational instability (NCI), was associated with nonlinear terms in the quasi-geostrophic equations

- The shortest wave can be presented in the grid has the wavelength $2\Delta x$ and computational wavenumber $$p_{\max} = \frac{2\pi}{L_{\min}} \Delta x = \pi$$



- Quadratic terms with Fourier components will generate higher wave numbers:

$$e^{\pm i p_1} e^{\pm i p_2} = e^{\pm i (p_1 \pm p_2)}$$

- The new shorter waves, with wave numbers $p=\pi+\delta$, cannot be represented in the grid, and become folded back (aliased) into $p'=\pi-\delta$, leading to a spurious accumulation of energy at the shortest range

An example of NCI effect

- We have PDE: $$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

and its corresponding FDE: $$\frac{\partial U_j}{\partial t} = -U_j \frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

Suppose at a given time t, we have:

$U_1=0$, $U_2>0$, $U_3<0$, $U_4=0$.

Then

$$\frac{\partial U_1}{\partial t} = 0, \frac{\partial U_2}{\partial t} > 0, \frac{\partial U_3}{\partial t} < 0, \frac{\partial U_4}{\partial t} = 0$$

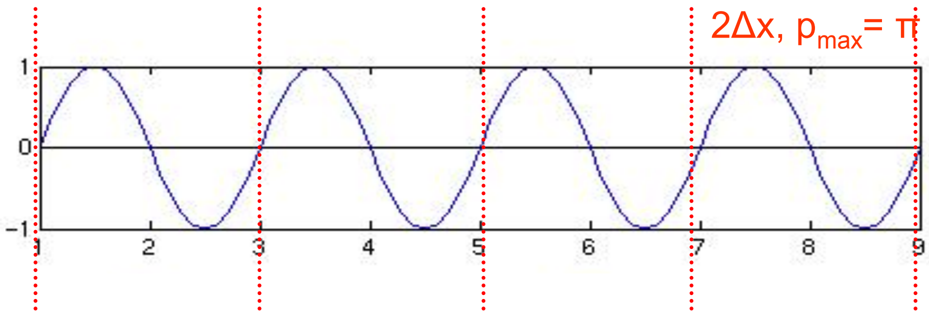
U2 and U3 will grow without bound and the FDE will blow up

this will happen even for a linear model $$\frac{\partial U_j}{\partial t} = -a_j \frac{U_{j+1} - U_{j-1}}{2\Delta x}$$

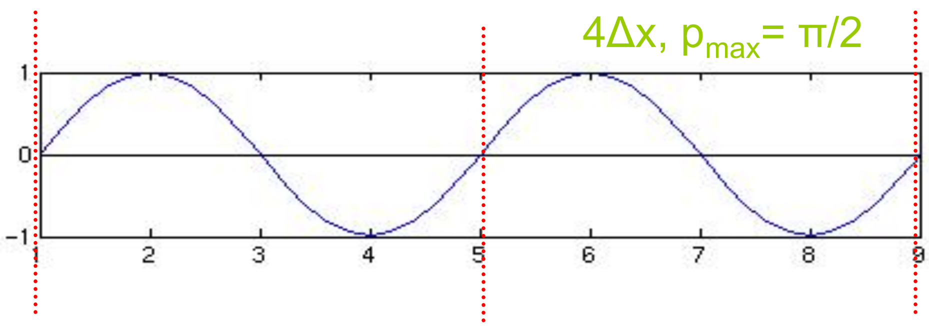
Approaches for dealing with NCI problem:
A) Filtering out high wavenumbers

1. Chop the high wavenumbers (wavenumbers between π/2 ~ π), Phillips (1959)

$P = k\Delta x = (L/2\pi)\,\Delta x$, $\quad e^{\pm i p_1} e^{\pm i p_2} = e^{\pm i (p_1 \pm p_2)}$



Originally, the Fourier transform wave number is (0~π), therefore the max wave number generated in a *quadratic term* can be p_{max}'=2π which can not be presented in the grid.



If chop half of the spectrum, the remained wave number is (0~π/2), the max wave number generated in a *quadratic term* is p_{max}'=π which is still can be presented in the grid

However the procedure is rather inefficient, since half of the spectrum is not used.

Approaches for dealing with NCI problem:
A) Filtering out high wavenumbers

However, for grid point models, experience shows that complete Fourier filtering of the high wave numbers is not necessary. Some models filter high wave numbers but only enough to maintain computational stability.

For example, Shapiro filter

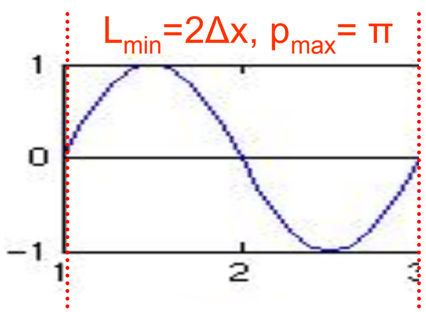
$$\overline{U}_j^{2n} = [1-(-D)^n]U_j$$

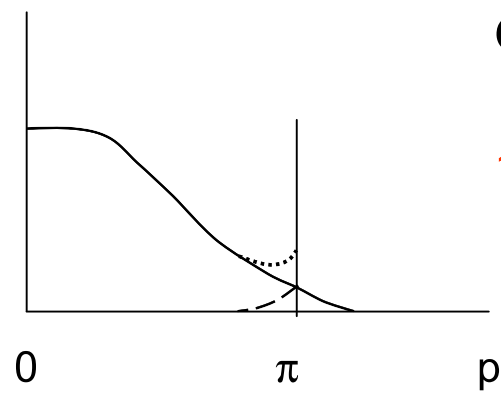
$$DU_j = (U_{j+1} - 2U_j + U_{j-1})/4$$

“diffusion” operator is applied to the original field n times. This efficiently filtered out the shortest waves (mostly between 2 and 3Δx) without affecting waves of wavelength 4Δx or longer, and resulted in an accurate and economic model

Approaches for dealing with NCI problem:
A) Filtering out high wavenumbers

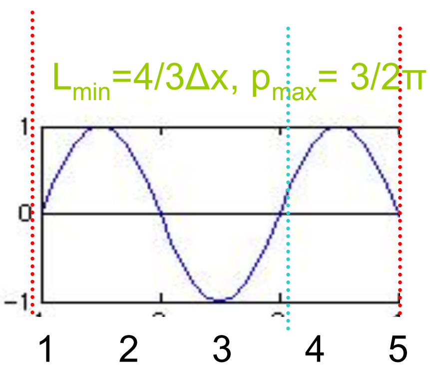
For spectral model, Orszag (1971) showed when transformed original spectrum back into grid points, if we use a grid with 3/2 as many grid points as the original grid, then the aliasing is avoided.

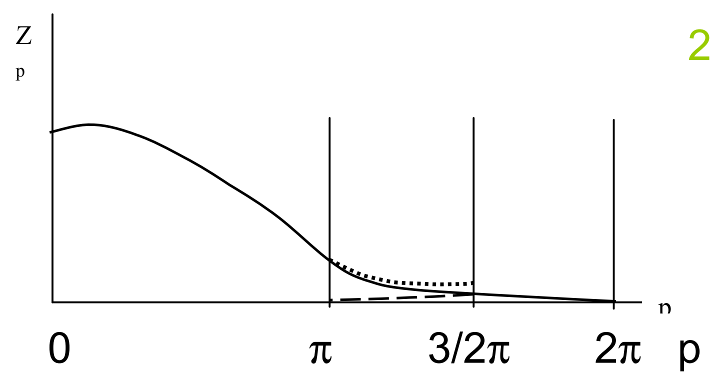




Original P_{max}= π, and the new P_{max}' from a quadratic product is 2π

1) If we do not filter high waveumbers, the grids can not handle the wavenumbers > π, all the waves above p= π are folded back.





2) If we filter above P=3/2 π, only the waves above P=3/2 π are folded back at 3/2 π, and 2 π gets folded back to π: no spurious alias below π

Approaches for dealing with NCI problem:
B) Using ***quadratically conserving schemes***

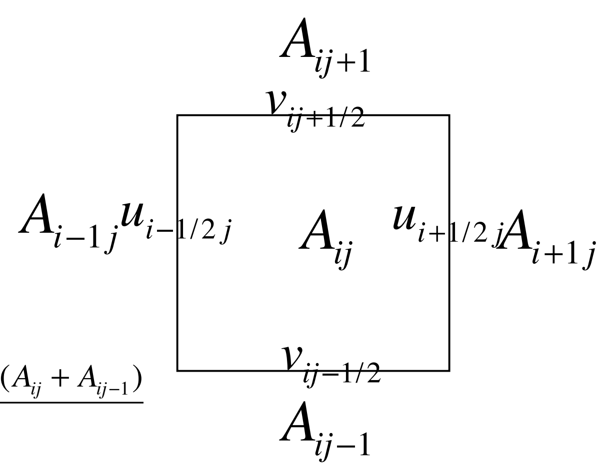
- Lilly (1965) showed that it is possible to create a spatial finite difference scheme that *conserves* both the mean value and its mean square value when integrated over a closed domain.

$$\frac{\partial}{\partial t}\sum_{i,j} A = 0$$

$$\frac{\partial}{\partial t}\sum_{i,j} A^2 = 0$$

Quadratic conservation will generally insure that NCI does not take place.

- The rule is that
  - We write the forecast equation for A consistently with the continuity equation
  - We estimate A at the walls of each cell as a simple average with the neighboring cell
  - For example, if the flow is nondivergent



$$\frac{u_{i+1/2j} - u_{i-1/2j}}{\Delta x} + \frac{v_{i-1/2j} - v_{i-1/2j}}{\Delta y} = 0$$

$$\frac{\partial A_{ij}}{\partial t} = -\frac{u_{i+1/2j}(A_{ij} + A_{i+1j}) - u_{i-1/2j}(A_{ij} + A_{i-1j})}{2\Delta x} - \frac{v_{i+1/2j}(A_{ij} + A_{ij+1}) - v_{i-1/2j}(A_{ij} + A_{ij-1})}{2\Delta y}$$

Then $\sum_{i,j} A_{i,j} \frac{\partial}{\partial t} A_{i,j} = 0$ because the fluxes at the walls cancel out: what leaves one cell enters the neighboring cell. Exercise: prove quadratic conservation

Let's start from SWE:

$$\frac{\partial(h\frac{\alpha^2}{2})}{\partial t} = \frac{\alpha^2}{2}\frac{\partial h}{\partial t} + h\alpha\frac{\partial \alpha}{\partial t} = -\frac{\alpha^2}{2}\frac{\partial h}{\partial t} + \alpha\frac{\partial h\alpha}{\partial t}$$

h $$\frac{\partial \alpha}{\partial t} + \mathbf{v}\nabla\alpha = 0$$

α $$\frac{\partial h}{\partial t} + \mathbf{v}\nabla h + h\nabla\mathbf{v} = \frac{\partial h}{\partial t} + \nabla h\mathbf{v} = 0$$

For PDE in flux form:

$$\frac{\partial h\alpha}{\partial t} + \nabla h\mathbf{v}\alpha = 0$$

$$\frac{\partial h}{\partial t} + \nabla h\mathbf{v} = 0$$

The mass weighted mean of α is conserved

$$\frac{\partial}{\partial t}\iint h\alpha dxdy = 0$$

$$\frac{\partial}{\partial t}\iint h dxdy = 0$$

$$\frac{\partial}{\partial t}\iint h\frac{\alpha^2}{2} dxdy = 0$$

The total mass is conserved in time

The mass weighted mean squared of α is conserved

For FDE in flux form:

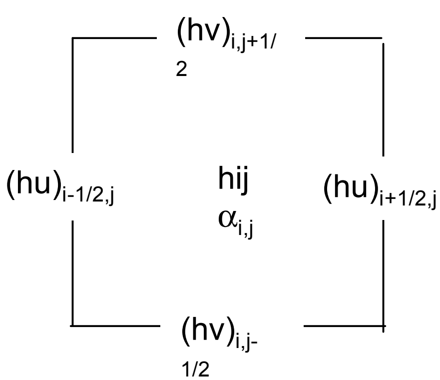
$$\frac{\partial h_{ij}}{\partial t} + \frac{(hu)_{i+1/2,j} - (hu)_{i-1/2,j}}{\Delta x} + \frac{(hv)_{i,j+1/2} - (hv)_{i,j-1/2}}{\Delta y} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial t}\sum_{i,j} h_{i,j}\Delta x_{i,j}\Delta y_{i,j} = 0}$$

$$\frac{\partial h_{ij}\alpha_{i,j}}{\partial t} + \frac{(hu)_{i+1/2,j}(\alpha)_{i+1/2,j} - (hu)_{i-1/2,j}(\alpha)_{i-1/2,j}}{\Delta x} + \frac{(hv)_{i,j+1/2}(\alpha)_{i,j+1/2} - (hv)_{i,j-1/2}(\alpha)_{i,j-1/2}}{\Delta y} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial t}\sum_{i,j} h_{i,j}\alpha_{i,j}\Delta x_{i,j}\Delta y_{i,j} = 0}$$

Because $$\frac{\partial\left(h\frac{\alpha^2}{2}\right)}{\partial t} = \frac{\alpha^2}{2}\frac{\partial h}{\partial t} + h\alpha\frac{\partial \alpha}{\partial t} = -\frac{\alpha^2}{2}\frac{\partial h}{\partial t} + \alpha\frac{\partial h\alpha}{\partial t}$$

And if we use $$(\alpha)_{i+1/2,j} = (\alpha_{i,j} + \alpha_{i+1,j})/2$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial t}\sum_{i,j} h_{i,j}\frac{1}{2}\alpha_{i,j}{}^2\Delta x_{i,j}\Delta y_{i,j} = 0}$$



We have quadratic conservation, and we could choose several FD formulations as long as,

1. the flux form of the FDE for $h\alpha$ is consistent with the continuity equation
2. we estimate α at the walls by a simple average

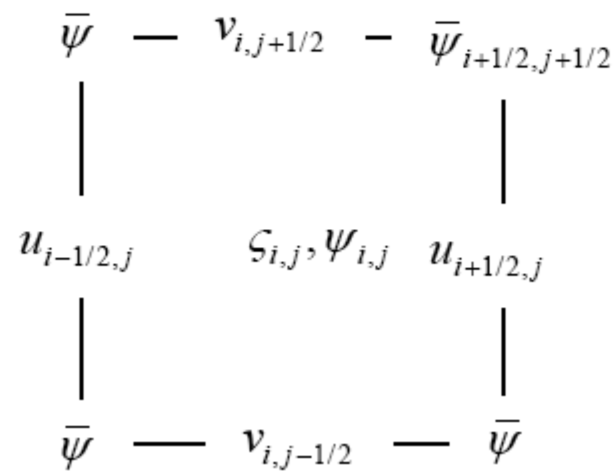
Finally consider the vorticity equation

<u>Its PDE in flux form:</u>

$$\frac{\partial \xi}{\partial t} = -\mathbf{v} \bullet \nabla \xi = -\nabla \bullet (\mathbf{v}\xi)$$

If we write its <u>FDE</u> in a way consistent with the continuity equation:

$$\frac{\partial \zeta_{i,j}}{\partial t} = -\frac{(u)_{i+1/2,j}(\zeta_{i,j} + \zeta_{i+1,j}) - (u)_{i-1/2,j}(\zeta_{i,j} + \zeta_{i-1,j})}{2\Delta x}$$
$$-\frac{(v)_{i,j+1/2}(\zeta_{i,j} + \zeta_{i,j+1}) - (v)_{i,j-1/2}(\zeta_{i,j} + \zeta_{i,j-1})}{2\Delta y}$$



This ensures conservation of the mean vorticity and enstrophy (mean square vorticity).

Steps for time integration:

1. Integrate vorticity equation from t_{n-1} to time step t_n
2. Calculate streamfunction at t from $\zeta = \nabla^2 \Psi$ (an elliptic equation)
3. Average to get streamfunction at corners
4. Calculate u,v at time step t from $u = -\frac{\partial \psi}{\partial y}; \quad v == \frac{\partial \psi}{\partial x}$
5. Forecast vorticity at next time step t_{n+1}