METO614

3.3.5, 3.3.6, and 3.4

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3.3.5 Staggered grids

SWE in two dimensions

$$\frac{\partial h}{\partial t} = -\left[h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \right] - u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y}$$

$$\frac{\partial u}{\partial t} = -\left[g\frac{\partial h}{\partial x} + fv \right] - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \qquad (0.19)$$

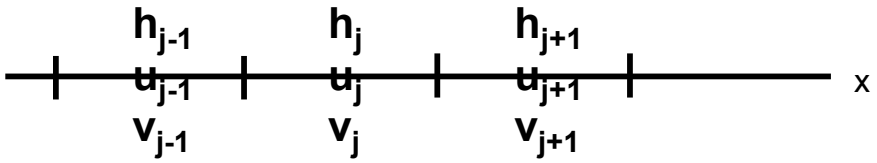
$$\frac{\partial v}{\partial t} = -\left[g\frac{\partial h}{\partial y} - fu \right] - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y}$$

- The term in brackets are the dominant terms for the geostrophic (2.5.1) and the inertia gravity (2.5.1) wave dynamics.
- These terms are computed in different ways depending on the type of grid we use.
- The advective terms are less affected by the choice of alternative (staggered) grids.

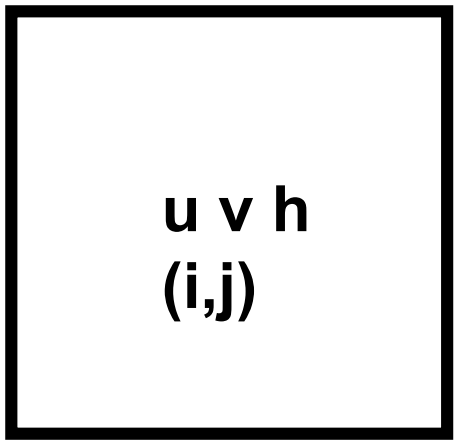
How to write the terms in brackets in differential (FDE) form??

Un-staggered grid or Staggered grid??

- **Un-staggered grid:**



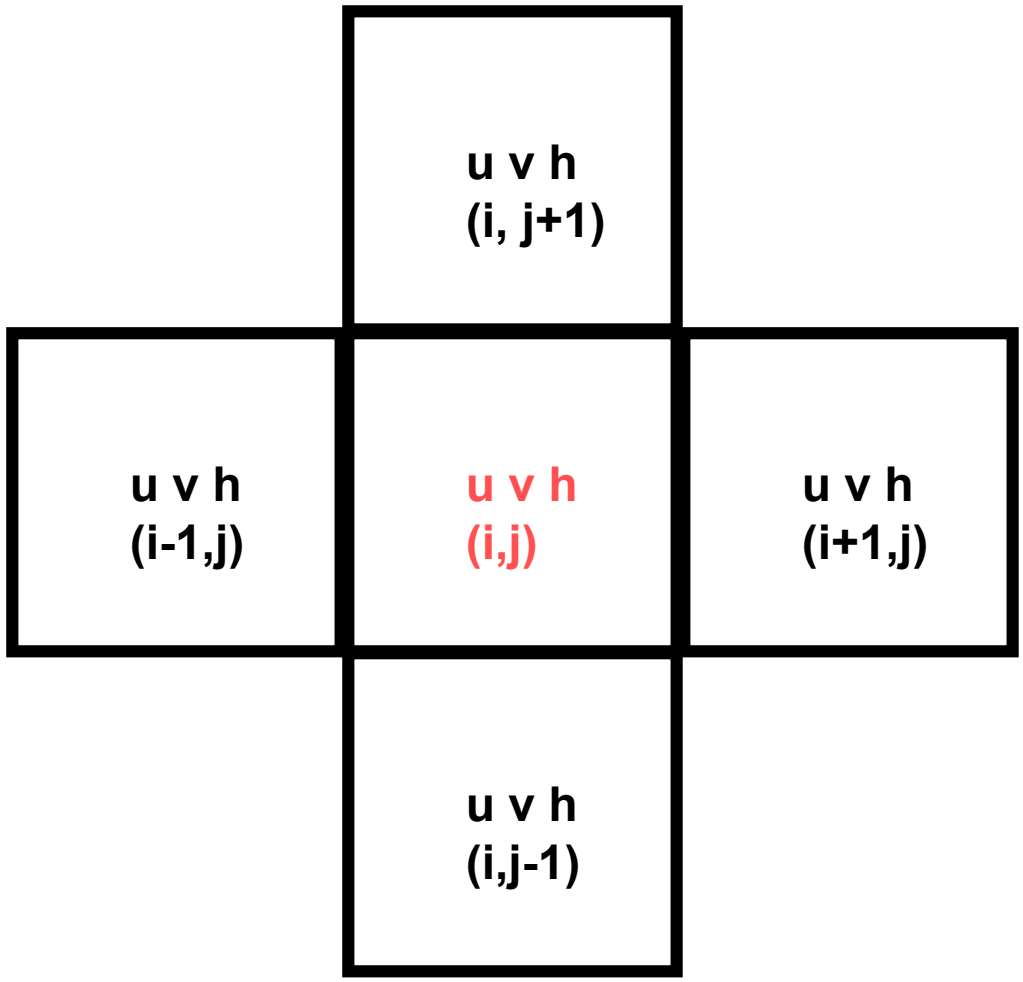
$$\delta_{2x}U_j = \frac{U_{j+1} - U_{j-1}}{2\Delta x} \cong U_x + \frac{\Delta x^2}{6}U_{xxx} + ...(3.3.1)$$



- Difference computed over a distance of 2d, while in a staggered grid it is over a distance of 1d

- All the variables are at the same location, so easy to implement higher order scheme.

Horizontal uncoupling in grid A



un-staggered grid A

- Geopotential height eq.

$$\frac{h_{i,j}^{n+1} - h_{i,j}^{n}}{2\Delta t} = -h_{i,j}^{n}\left(\frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta x} + \frac{v_{i,j+1}^{n} - v_{i,j-1}^{n}}{2\Delta y}\right) + advection$$

the geopotential height is not coupled with the wind at the point (i,j)

- Zonal wind eq.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} = -g \times \frac{h_{i+1,j}^{n} - h_{i-1,j}^{n}}{2\Delta x} + f \times v_{i,j}^{n} + advection$$

U is not coupled with h at point (i,j) either

Un-staggered grid A

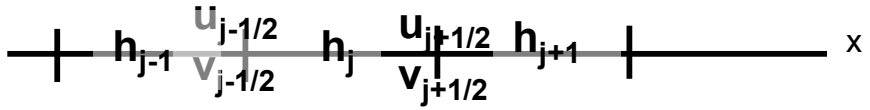
Advantage:

- **Because all variables are available at all the grid points, it is easy to construct a higher order accuracy scheme.**
- **Grid A tends to be favored by proponents of the philosophy "accuracy is more important than conservation".**

Disadvantage:

- **Its main disadvantage is that all differences occur on distances 2d**
- **Neighboring points are not coupled for the pressure and convergence terms. This can give rise in time to a horizontal uncoupling (checkerboard pattern),**

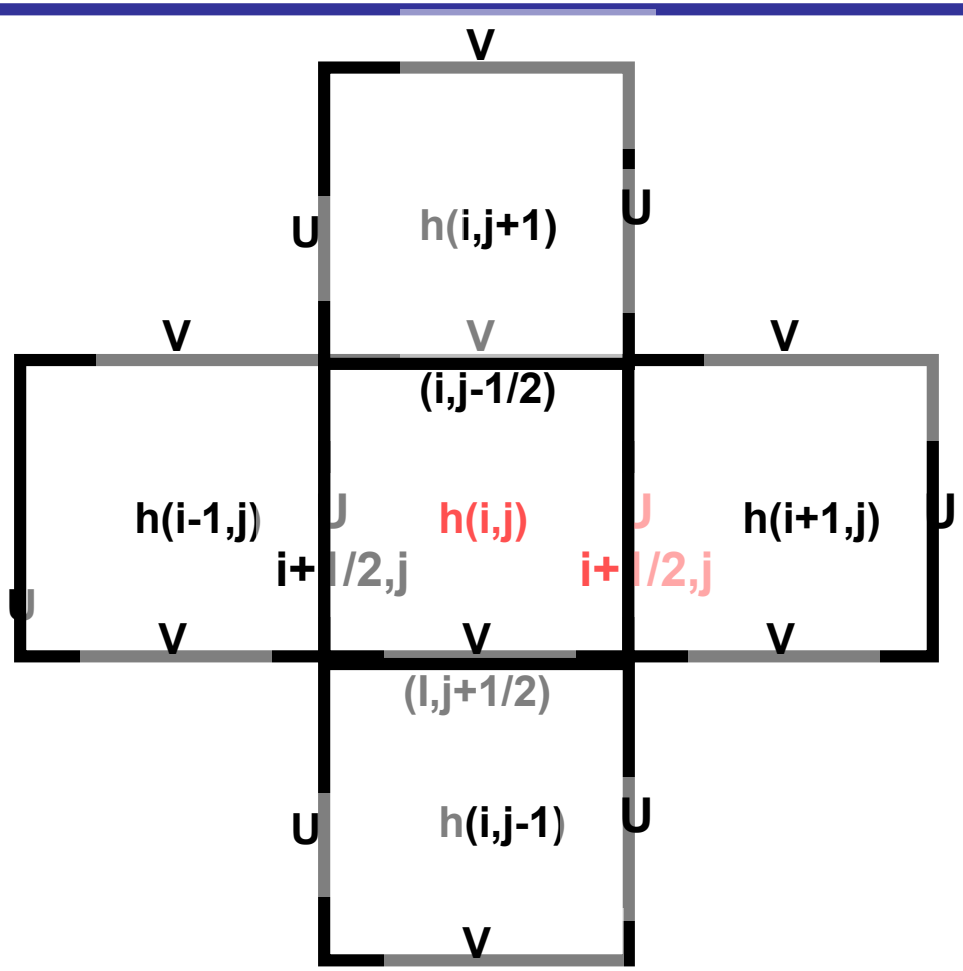
Staggered grid:



❖staggered grid happens over a distance of 1d

$$\delta_x U_j = \frac{U_{j+\frac{1}{2}} - U_{j-\frac{1}{2}}}{\Delta x} \cong U_x + \frac{1}{4} \times \frac{\Delta x^2}{6} U_{xxx} + ...$$

Types of staggered grid:
1) staggered grid C



1) staggered grid C

- Geopotential height eq.

$$\frac{h_{i,j}^{n+1} - h_{i,j}^{n-1}}{2\Delta t} = -h_{i,j}^{n}\left(\frac{u^{n}_{i+\frac{1}{2},j} - u^{n}_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^{n}_{i,j+\frac{1}{2}} - v^{n}_{i,j-\frac{1}{2}}}{\Delta y}\right) + advection$$

- Zonal wind eq.

$$\frac{u^{n+1}_{i+\frac{1}{2},j} - u^{n-1}_{i+\frac{1}{2},j}}{2\Delta t} = -g \times \frac{h^{n}_{i+1,j} - h^{n}_{i,j}}{\Delta x} + f \times \bar{v}^{n}_{i+\frac{1}{2},j} + advection$$

$$\bar{v}^{n}_{i+\frac{1}{2},j} = \frac{v^{n}_{i,j-\frac{1}{2}} + v^{n}_{i,j+\frac{1}{2}} + v^{n}_{i+1,j-\frac{1}{2}} + v^{n}_{i+1,j+\frac{1}{2}}}{4}$$

1) staggered grid C

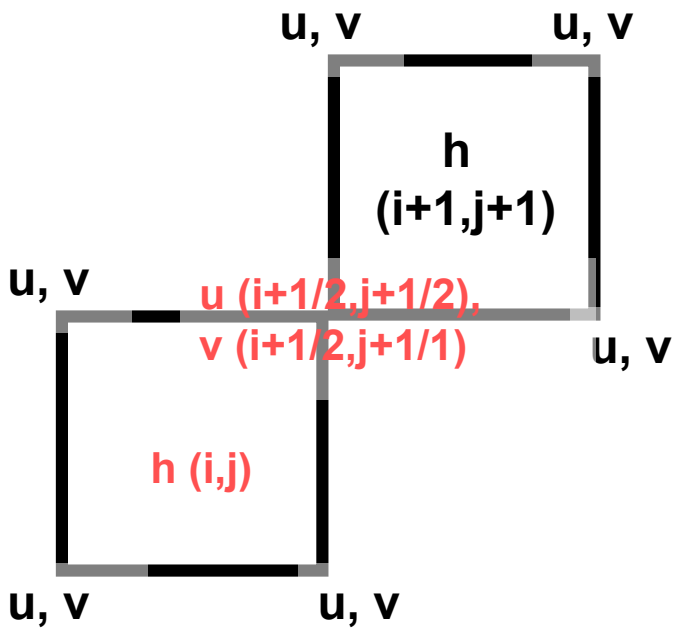
Advantage:

- **The convergence and pressure terms are computed over a distance of only 1d, equivalent *to doubling the resolution of grid A***

Disadvantage:

- **The Coriolis acceleration terms, on the other hand, requires horizontal averaging, making the inertia gravity waves (related with Coriolis force) less accurate**
- **This makes the grid C less attractive for situations in which the length of the Rossby radius of deformation $R_d = \sqrt{gH} / f$ is not large compared to the grid size d.**

2) Staggered grid B



Staggered grid B

- Geopotential height eq.

$$\frac{h_{i,j}^{n+1}-h_{i,j}^{n}}{2\Delta t}=-h_{i,j}^{n}\left(\frac{u^{n}_{i+\frac{1}{2},j+\frac{1}{2}}-u^{n}_{i-\frac{1}{2},j-\frac{1}{2}}}{\sqrt{2}\Delta x}+\frac{v^{n}_{i+\frac{1}{2},j+\frac{1}{2}}-v^{n}_{i-\frac{1}{2},j-\frac{1}{2}}}{\sqrt{2}\Delta y}\right)+advection$$

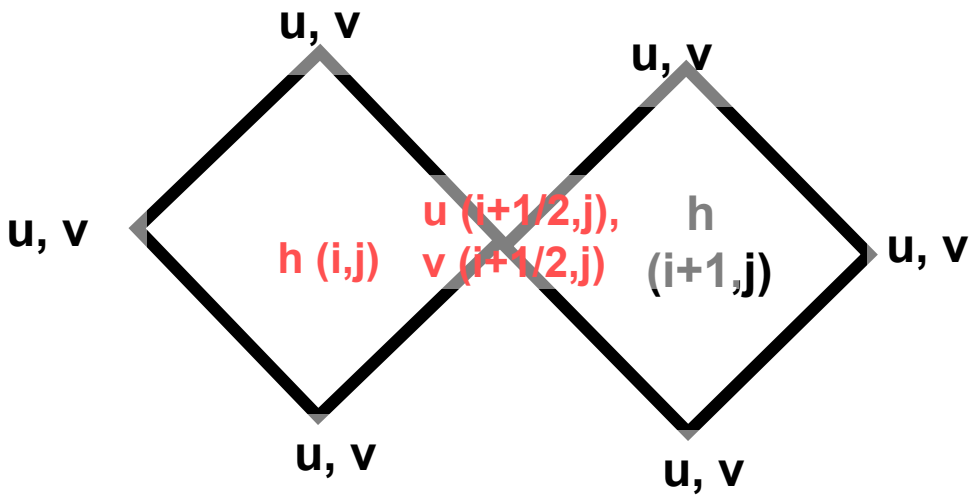
- Zonal wind eq.

$$\frac{u^{n+1}_{i+\frac{1}{2},j}-u^{n-1}_{i+\frac{1}{2},j}}{2\Delta t}=-g\times\frac{h^{n}_{i+1,j+1}-h^{n}_{i,j}}{\sqrt{2}\Delta x}+f\times v^{n}_{i+\frac{1}{2},j}+advection$$

Advantage: unnecessary to do averaging to calculate Coriolis force

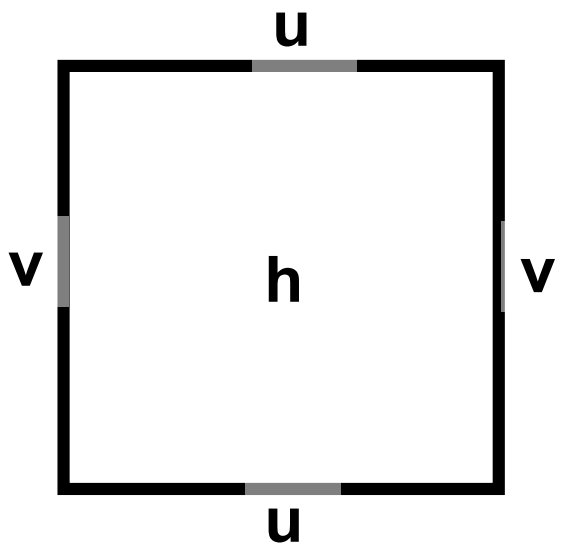
Disadvantage: the minimum distance for horizontal differences is $\sqrt{2}d$ rather than $1d$ as in grid C,

Staggered grid E



- The distance is still $\sqrt{2}d$ The NCEP Eta model is defined on a grid B rotated by 45, denoted grid E by Arakawa and Lamb (1977),
- Only the index is different with grid B.

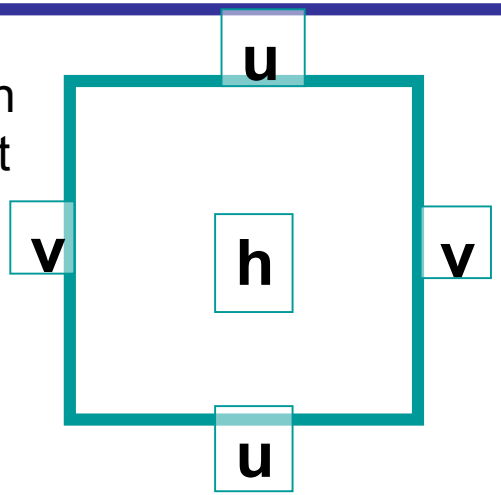
3) Staggered grid D



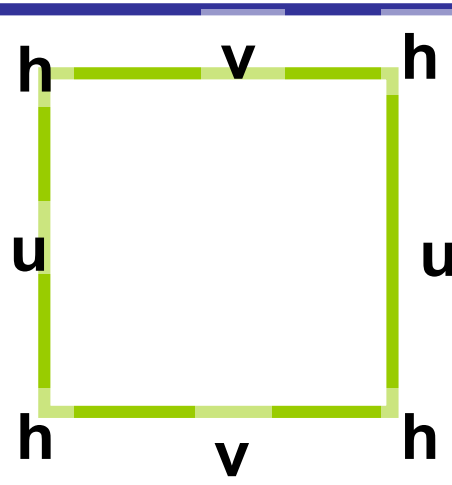
- **With u, v exchanges the location, grid C becomes grid D.**
- **Grid D has no particular merit;**
- **When staggered in time (as suggested by Eliassen), it becomes ideal for atmospheric flow using the Leap-Frog Scheme (see Fig. 3.14)**

4) The Eliassen Grid

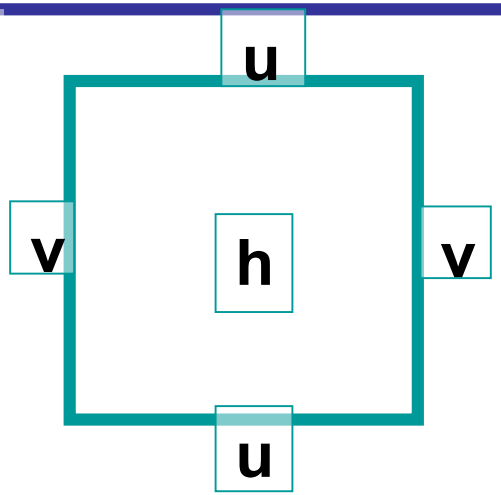
Geopotential height eq.



Even time steps (n-1)

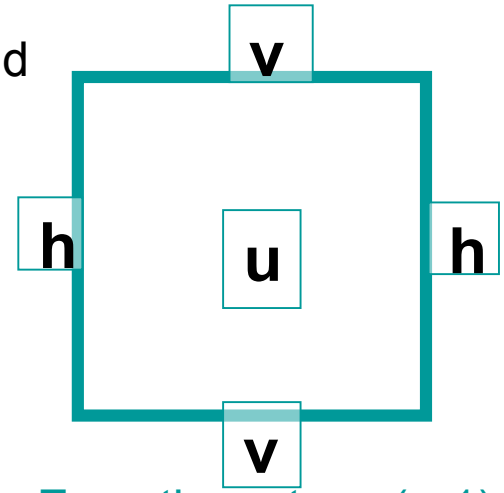


Odd time steps (n)

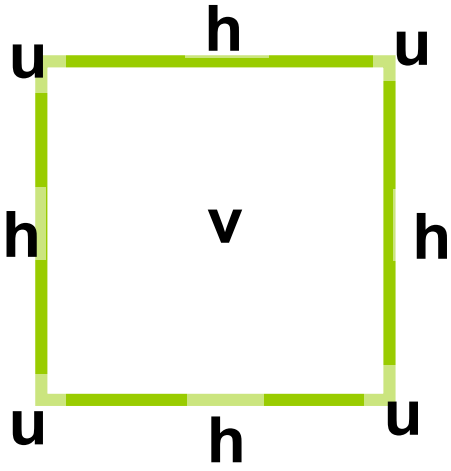


Even time steps (n+1)

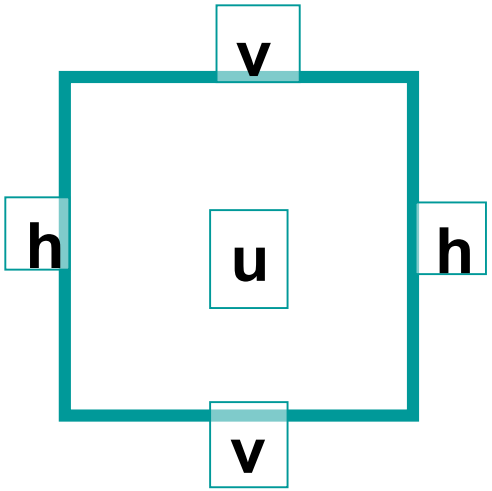
Zonal wind eq.



Even time steps (n-1)



Odd time steps (n)



Even time steps (n+1)

The Eliassen Grid

- Geopotential height eq.

$$\frac{h_{i,j}^{n+1} - h_{i,j}^{n-1}}{2\Delta t} = -h_{i,j}^{n}\left(\frac{u^{n}_{i+\frac{1}{2},j} - u^{n}_{i-\frac{1}{2},j}}{\Delta x} + \frac{v^{n}_{i,j+\frac{1}{2}} - v^{n}_{i,j-\frac{1}{2}}}{\Delta y}\right) + advection$$

- Zonal wind eq.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n-1}}{2\Delta t} = -g \times \frac{h^{n}_{i+\frac{1}{2},j} - h^{n}_{i-\frac{1}{2},j}}{\Delta x} + f \times v_{i,j}^{n} + advection$$

Advantage: almost perfect, the divergence term and pressure gradient are calculated over 1d.

❖However, it has not been adopted in any major model because of complexity of time staggering… Should be tried?

Vertical differential equation
e.g. Quasi-Geostrophic equation

$$\left(\frac{\partial}{\partial t} + v_g \bullet \nabla\right)\left(\nabla^2 \psi + f\right) - f_0 \frac{\partial \omega}{\partial p} = 0$$

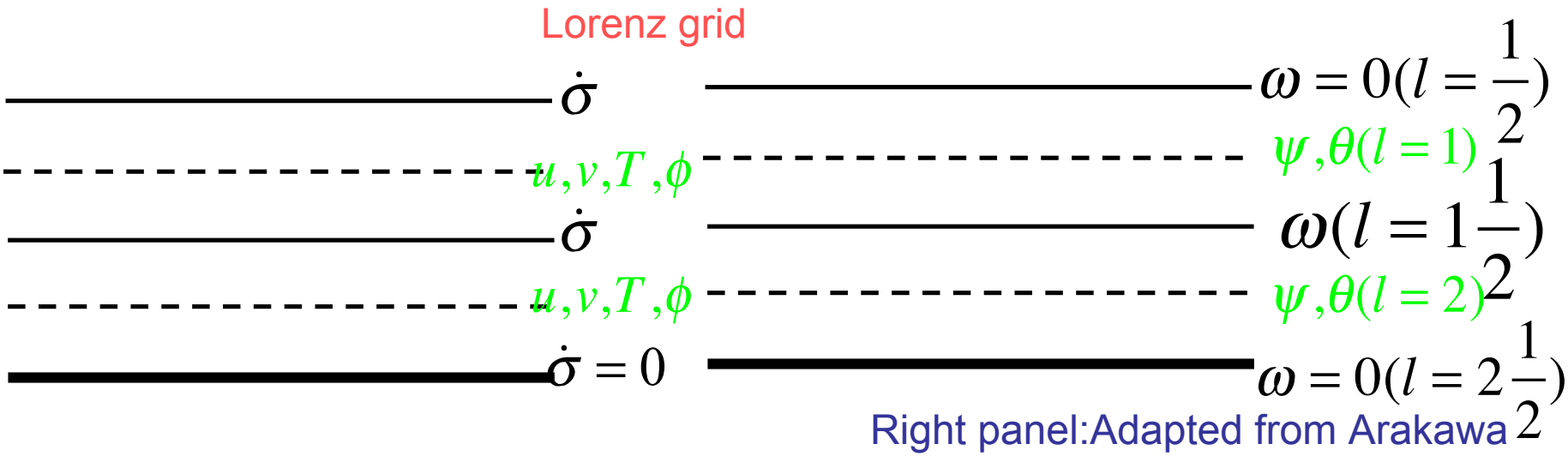
$$\left(\frac{\partial}{\partial t} + v_g \bullet \nabla\right)\frac{\partial \psi}{\partial p} + \frac{\omega}{f_0} S = 0$$

$$S \equiv -(\alpha / \theta)\frac{d\bar{\theta}}{dp}$$

Arakawa and Moorthi, 1988

How to write FDE for the terms in green?

Vertical staggering method



Right panel:Adapted from Arakawa and Moorthi (1988)

$$\frac{\partial\omega}{\partial p}(l=1)=\frac{\omega(l=1\frac{1}{2})-\omega(l=\frac{1}{2})}{\triangle p}$$

$$S=-(\alpha/\bar{\theta})\frac{d\bar{\theta}}{dp}$$

$$\frac{\partial\psi}{\partial p}(l=1\frac{1}{2})=\frac{\psi(l=2)-\psi(l=1)}{\triangle p}$$

$$\bar{\theta}(l=1\frac{1}{2})=\frac{\theta(l=2)+\theta(l=1)}{2}$$

Lorenz grid

❖ Problem: Allows the development of a spurious computational mode

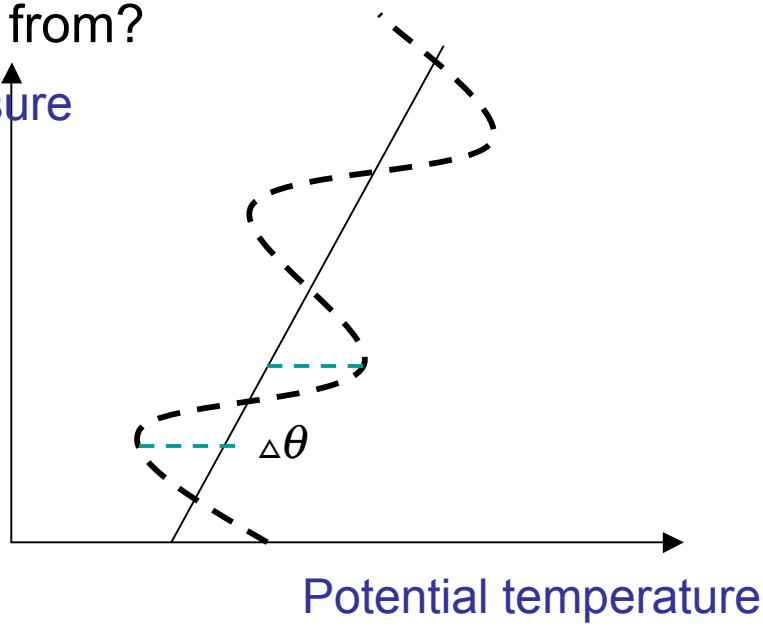
❖ Reason: The existence of an extra degree of freedom in potential temperature

Where does this extra freedom come from?

$$\frac{\partial \phi}{\partial (p / p_0)^k} = -c_p \theta$$

$$\phi_l - \phi_{l+1} = c_p \theta_{l+\frac{1}{2}} [(p_{l+1} / p_0)^k - (p_l / p_0)^k]$$

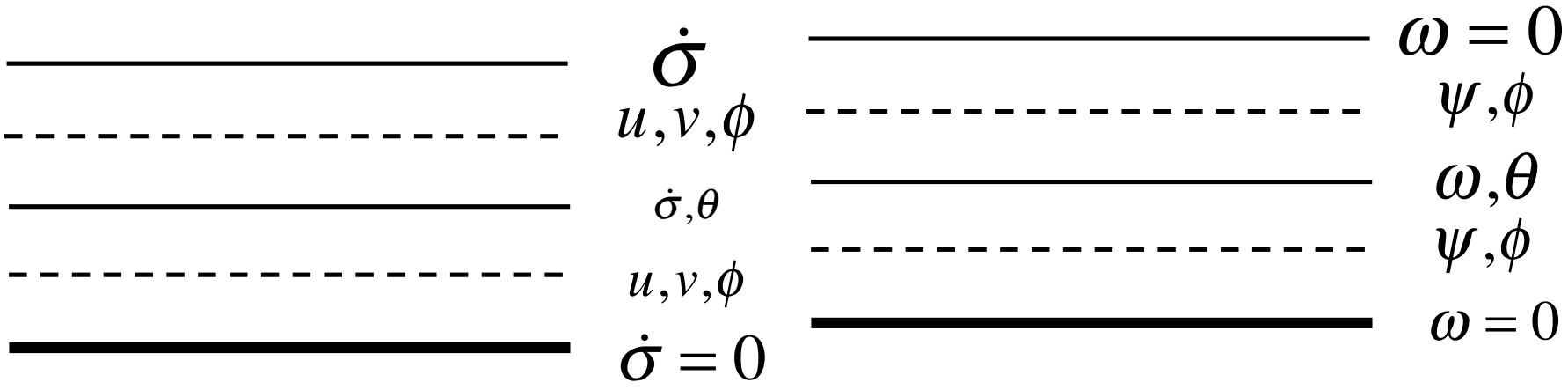
$$\theta_{l+\frac{1}{2}} = \frac{1}{2}(\theta_l + \theta_{l+1})$$



$\theta_{l+\frac{1}{2}}$ is not affected by superposing $(-1)^l \Delta\theta$

Produces a zig-zag vertical profile of potential temperature

Charney-Phillips grid

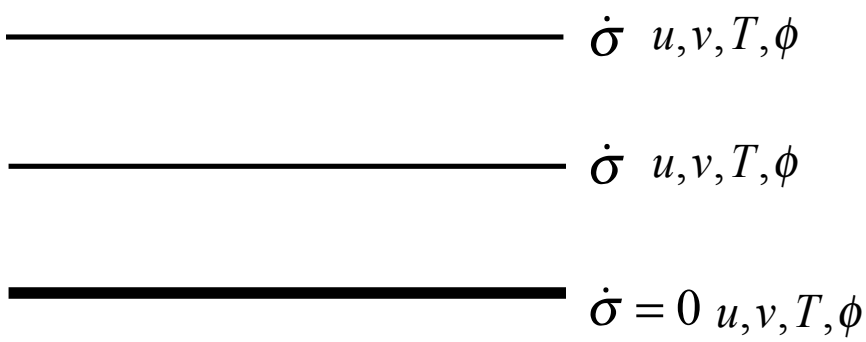


$$\frac{\partial \phi}{\partial (p / p_0)^k} = -c_p \theta$$

$$\phi_l - \phi_{l+1} = c_p \theta_{l+\frac{1}{2}} [(p_{l+1} / p_0)^k - (p_l / p_0)^k]$$

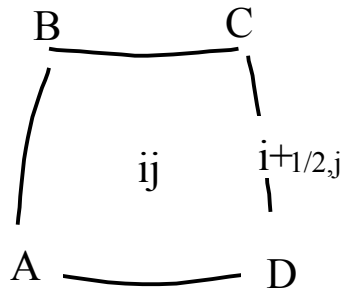
$\theta_{l+\frac{1}{2}}$ is not calculated from averaging anymore, so there is no spurious computational mode

Un-staggered vertical grid



Introduce computational mode

3.3.6 Finite Volume methods



➢The basic idea: The governing equations are first written in an integral form for a finite volume, and only then they are discretized.

➢This is in contrast to the methods we have seen so far, in which the equations in differential form are discretized using finite differences or spectral methods

Example: continuity equation and mass weighted equation

$$\frac{\partial h\alpha}{\partial t} + \nabla \bullet h\mathbf{v}\alpha = 0 \qquad \frac{\partial h}{\partial t} + \nabla \bullet h\mathbf{v} = 0$$

$$\Rightarrow$$

$$\frac{d}{dt}\int h dx dy + \oint \mathbf{H}\bullet\mathbf{n} ds = 0 \qquad \frac{d}{dt}\int h\alpha dx dy + \oint (\mathbf{H}\alpha)\bullet\mathbf{n} ds = 0$$

where **H** is the normal flux of *h* across the walls, and **n** is the normal vector to the wall.

Differential form α is any other variable: u, v, T

$$\frac{d}{dt}\left(\overline{h}^{ij}\Delta x_{ij}\Delta y_{ij}\right) = -(\overline{hu}^{i+1/2\,j})\Delta y_{i+1/2\,j} + (\overline{hu}^{i-1/2\,j})\Delta y_{i-1/2\,j}$$

$$-(\overline{hv}^{ij+1/2})\Delta x_{ij+1/2} + (\overline{hv}^{ij-1/2})\Delta x_{ij-1/2}$$

$$\frac{d}{dt}\left(\overline{\alpha h}^{ij}\Delta x_{ij}\Delta y_{ij}\right) = -(\overline{hu\alpha}^{i+1/2\,j})\Delta y_{i+1/2\,j} + (\overline{hu\alpha}^{i-1/2\,j})\Delta y_{i-1/2\,j}$$

$$-(\overline{hv\alpha}^{ij+1/2})\Delta x_{ij+1/2} + (\overline{hv\alpha}^{ij-1/2})\Delta x_{ij-1/2}$$

- The overbar indicates a suitable average over the grid volume or area
- Any scheme based on these finite-volume equations will conserve the average mass and average mass weighted value of α

Characteristics of finite volume method

- There are a number of choices on how this average can be carried out over this *subgrid* domain of each grid volume.
- one can assume that *h* or α are constant within the volume, or that they vary linearly.
- A simple choice for the estimates of the average values at the center and at the walls leads naturally to the quadratically conservative differences (3.3.4)

$$\bar{h}^{ij} = h_{ij}; \quad \overline{hu}^{i+1/2\,j} = (h_{ij} + h_{i+1\,j})(u_{ij} + u_{i+1\,j}) / 4;$$

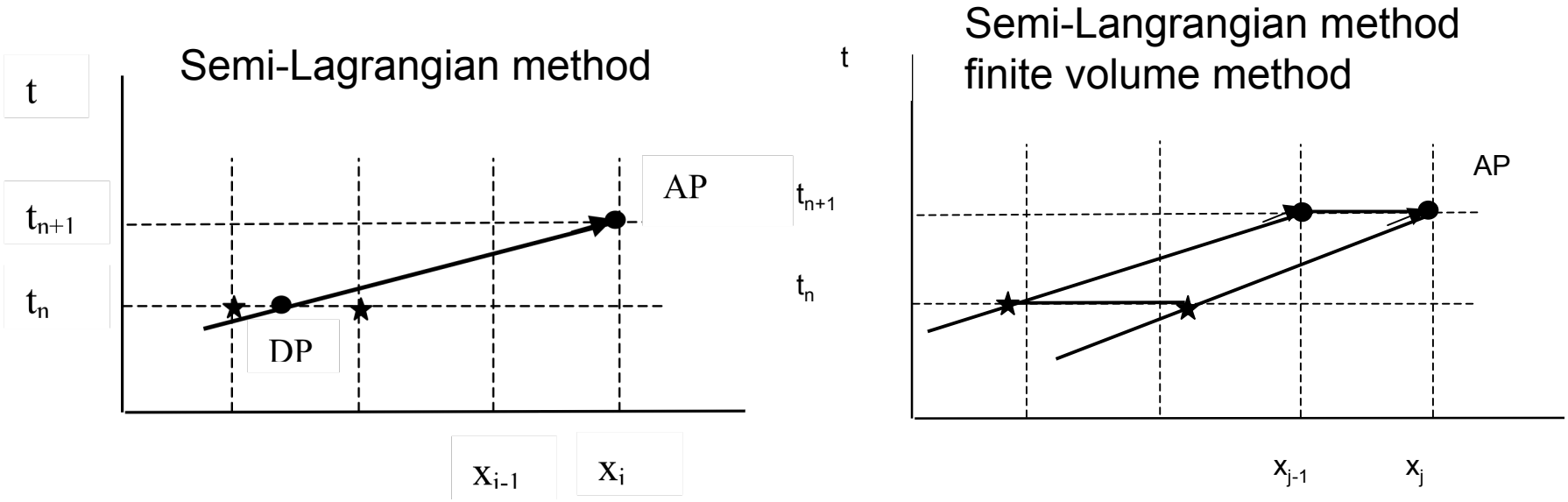
$$\overline{hu\alpha}^{i+1/2\,j} = (\overline{hu}^{i+1/2\,j})(\alpha_{ij} + \alpha_{i+1\,j}) / 2$$

Finite volume allows additional flexibility in the choice of discretization.

Example: semi-Lagrangian finite volume method: Lin and Rood (1996)

This approach has been adopted in several NCAR, NASA and DOE global community models because of its conservation properties

Semi-Lagrangian method and semi-Lagrangian finite volume method



- The boundaries of the grid volume are transported to the new time step, rather than the centers of the volume as is done in the conventional semi-Lagrangian schemes
- It requires considerable care in the detailed formulation in order to remain both conservative and maintain the shape of the transported tracers. To learn detail, check Lin and Rood (1996)
- Advantages: Conservative, accurate and no constraint on time step for advection!

3.4 Boundary value problems

3.4.1 Introduction

Elliptic equations are boundary value problems (3.1.1)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0(f(x,y))$$

U can be any variable.

To solve this equation, we need two boundary conditions, one on each x and y boundary

3.4.1 Introduction

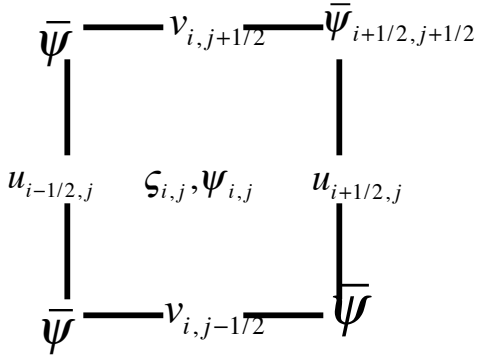
Two examples of such problems arising in NWP.

a) Finding the new stream function from the vorticity after the latter has been updated to time (n+1)Δt. Enstrophy conserving numerical scheme:

Process

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v}\bullet\nabla\zeta = -\nabla\bullet(\mathbf{v}\zeta) = -J(\Psi,\zeta)$$

C-grid



$$\frac{\zeta_{i,j}^{n+1} - \zeta_{i,j}^{n-1}}{2\Delta t} = -\frac{(u)_{i+1/2,j}(\zeta_{i,j} + \zeta_{i+1,j}) - (u)_{i-1/2,j}(\zeta_{i,j} + \zeta_{i-1,j})}{2\Delta x} - \frac{(v)_{i,j+1/2}(\zeta_{i,j} + \zeta_{i,j+1}) - (v)_{i,j-1/2}(\zeta_{i,j} + \zeta_{i,j-1})}{2\Delta y} \quad (3.4.1)$$

3.4.1 Example 1

$$\zeta = \mathbf{k}\bullet\nabla\times\mathbf{v} = \nabla^2\Psi, \quad \mathbf{v} = \mathbf{k}\times\nabla\Psi$$

After solving for $\zeta_{i,j}^{n+1}$, we can obtain the streamfunction by solving the elliptic equation (Laplace) valid at $t = (n+1)\Delta t$

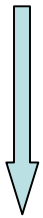
$$\frac{\Psi_{i+1,j} - 2\Psi_{i,j} + \Psi_{i-1,j}}{\Delta x^2} + \frac{\Psi_{i,j+1} - 2\Psi_{i,j} + \Psi_{i,j-1}}{\Delta y^2} = \zeta_{i,j} \qquad (3.4.2)$$

For this particular scheme, after solving for $\zeta_{i,j}^{n+1}$, we obtain by averaging from the four surrounding corners $\overline{\Psi}_{i+1/2,j+1/2}^{n+1}$

Example 2

b) Solving a semi-implicit elliptic equation for the heights also at (n+1) Δt (section 3.2.5):

$$[\delta_{2x}^2+\delta_{2y}^2-\frac{1}{\Phi\Delta t^2}]\phi^{n+1}=-[\delta_{2x}^2+\delta_{2y}^2+\frac{1}{\Phi\Delta t^2}]\phi^{n-1}+2[\delta_{2x}R_u+\delta_{2y}R_v]$$
$$+\frac{1}{\Delta t}\left[\delta_{2x}u^{n-1}+\delta_{2y}v^{n-1}\right]-\frac{2}{\Phi\Delta t}R_\phi=F_{i,j}^n \qquad (3.4.3)$$



$$\delta^2\phi_{i,j}-\alpha\phi_{i,j}=g_{i,j}$$

Example 2

Directly solving the equation by spectral methods is easy:

For example, if we use spherical harmonics on the globe, and make use of

$$\nabla^2 Y_n^m = \frac{1}{a^2}\left[\frac{1}{\cos^2\varphi}\frac{\partial^2 Y_n^m}{\partial\lambda^2} + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial Y_n^m}{\partial\varphi}\right)\right] = \frac{-n(n+1)}{a^2}Y_n^m \qquad (4.4)$$

Suppose: $$\phi^{n+1}(\lambda,\varphi,t) = \sum_{n=0}^{N}\sum_{m=-n}^{n}\phi_n^m(t)Y_n^m(\lambda,\varphi) \qquad (4.5)$$

so that the solution for each spherical harmonic coefficient is given by :

$$\phi_m^n(t_{p+1}) = -\frac{1}{\left(\frac{n(n+1)}{a^2} + \frac{1}{\Phi\Delta t^2}\right)}F_m^n(t_{p+1}) \qquad (4.6)$$

3.4.2 Direct methods for linear system

Direct methods involve solving equations like (3.4.2) or (3.4.3), which can be written in matrix form as:

$$\mathbf{A}\phi = F \tag{3.4.7}$$

Soved by Gaussian elimination.

- If the matrix A is fixed (e.g., independent of the time step), the LU decomposition of A=LU
- solve LX=F
- followed by UΦ=X.

Direct method: e.g, double Sweep method

Problem: If the matrix is A tridiagonal, the direct problem is particularly easy to solve. A tridiagonal problem can be written as:

$$\begin{pmatrix} a_{j-1} & b_{j-1} & 0 \\ a_j & b_j & c_j \\ 0 & a_{j+1} & b_{j+1} \end{pmatrix} \begin{pmatrix} U_{j-1} \\ U_j \\ U_{j+1} \end{pmatrix} = \begin{pmatrix} d_{j-1} \\ d_j \\ d_{j+1} \end{pmatrix}$$

$$a_j U_{j-1} + b_j U_j + c_j U_{j+1} = d_j \tag{3.4.8}$$

with general boundary conditions

$$U_0 = A_1 U_1 + A_2, \quad U_J = B_1 U_{J-1} + B_2 \tag{3.4.9}$$

double Sweep method

Boundary condition: $U_0 = A_1 U_1 + A_2, \quad U_J = B_1 U_{J-1} + B_2$

Assume $U_j = E_j U_{j+1} + F_j$

Then $U_{j-1} = E_{j-1} U_j + F_{j-1}$ (3.4.10)

Substitute into (3.4.8), obtain:

$$(a_j E_{j-1} + b_j) U_j + c_j U_{j+1} = d_j - a_j F_{j-1} \quad (3.4.11)$$

$$U_j = -\frac{c_j}{a_j E_{j-1} + b_j} U_{j+1} + \frac{d_j - a_j F_{j-1}}{a_j E_{j-1} + b_j}$$

$$E_j = \frac{-c_j}{a_j E_{j-1} + b_j} \quad (3.4.12)$$

$$F_j = \frac{d_j - a_j F_{j-1}}{a_j E_{j-1} + b_j}$$

double Sweep method

So the method of solution is

a)use the lower BC $U_0 = A_1 U_1 + A_2$ to determine $E_0 = A_1, F_0 = A_2$

b) sweep forward using (4.12) to obtain $E_j, F_j, \quad j = 1, \ldots J-1$

c) determine U_J, U_{J-1} from $U_{J-1} = E_{J-1} U_J + F_{J-1}$

and the upper BC

$$U_J = B_1 U_{J-1} + B_2$$

$$U_{J-1} = E_{J-1}(B_1 U_{J-1} + B_2) + F_{J-1}$$

d) determine $U_j, j = J-2, \ldots 1$ using (3.4.10)

3.4.3 iterative methods to solve elliptic equations

Basic idea

$\mathbf{A}\phi = F$ It can be written as:

$\phi = (I - A)\phi + F$, or

$$\phi = M\phi + F \qquad (3.4.13)$$

choosing an initial guess ϕ^0 then iterating (3.4.13)
The method converges if the spectral radius $\sigma(M) = \max|\lambda_i| < 1$

- Compared with the direct method, the iterative method may more economical in memory requirements of a computer
- Can solve sets of nonlinear equations.

Jacobi simultaneous relaxation method

Elliptical equation (3. 4. 3) can be written as: $\delta^2\phi_{i,j} - \alpha\phi_{i,j} = g_{i,j}$

Suppose we are in iteration . Then

$$\delta^2\phi_{i,j} = (\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j}) \quad (3.4.16)$$

$$\delta^2\phi^{\nu}_{i,j} - \alpha\phi^{\nu}_{i,j} = g_{i,j} + \in_{i,j} \quad (3.4.17)$$

$$\phi^{\nu+1}_{i,j} = \phi^{\nu}_{i,j} + \delta\phi^{\nu}_{i,j} \quad (3.4.\ 18)$$

$\delta\phi^{\nu}_{i,j}$ Is to make $\in_{i,j} = 0$ that's $\delta^2\phi^{\nu+1}_{i,j} - \alpha\phi^{\nu+1}_{i,j} = g_{i,j}$

Substitute (3.4.18) into above equation:

$$\delta^2\phi^{\nu+1} - \alpha(\phi^{\nu}_{i,j} + \delta\phi^{\nu}_{i,j}) = g_{i,j} \qquad \delta^2\phi^{\nu+1} = \delta^2\phi^{\nu} - 4\delta\phi^{\nu}_{i,j} \longrightarrow$$

$$\delta\phi^{\nu}_{i,j} = \frac{\delta^2\phi^{\nu}_{i,j} - \alpha\phi^{\nu}_{i,j} - g_{i,j}}{4+\alpha} \longrightarrow \qquad \phi^{\nu+1}_{i,j} = \phi^{\nu}_{i,j} + \frac{\Delta\phi^{\nu}_{i,j} - \alpha\phi^{\nu}_{i,j} - g_{i,j}}{4+\alpha}$$

Gauss-Seidel relaxation method

$$\phi_{i,j}^{\nu+1} = \phi_{i,j}^{\nu} + \frac{\phi_{i-1,j}^{\nu+1} + \phi_{i+1,j}^{\nu} + \phi_{i,j-1}^{\nu+1} + \phi_{i,j+1}^{\nu} - \alpha\phi_{i,j}^{\nu} - g_{i,j}}{4+\alpha} \qquad (3.4.\ 20)$$

Successive Over-relaxation method (SOR)

Change the sign $\in_{i,j}^{\nu+1}$ rather than making it equal to zero

$$\phi_{i,j}^{\nu+1} = \phi_{i,j}^{\nu} + \omega\frac{\phi_{i-1,j}^{\nu+1} + \phi_{i+1,j}^{\nu} + \phi_{i,j-1}^{\nu+1} + \phi_{i,j+1}^{\nu} - \alpha\phi_{i,j}^{\nu} - g_{i,j}}{4+\alpha}, \quad \text{with} \quad 1<\omega<2$$

Summary of these three methods

- Relaxation method sets the residual equal to zero in one of the equations, and then solves the value iteratively.
- Jacobi iteration only use the values from last iteration in the right hand side of the equation
- Gauss-Seidel iteration use the values from the current iteration after the values have updated
- Over-relaxation method over-corrects: make the residuals have opposite sign with the last time (SOR)

Covergence speed (from slowest to fastest):

Jacobi iteration; Gauss-Seidel iteration; Succesive Over Relaxation (SOR)

Other iterative methods

1. Alternating Direction Implicit (ADI)

$$\frac{\partial u}{\partial t} = \sigma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad (4.\ 22)$$

$$\frac{u^* - u^n}{\Delta t} = \sigma \delta_x{}^2 u^*$$

$$\frac{u^{n+1} - u^*}{\Delta t} = \sigma \delta_y{}^2 u^{n+1} \qquad (4.\ 23)$$

Since each fractional step is implicit, large time steps can be used. And since the solution of each fractional step involves only inverting tridiagonal matrices, it can be performed very efficiently (see, e.g., Hageman and Young, 1981).

2. Multigrid methods

Reason to use multigrid: the iterative schemes depends on the number of grid points, and is much faster for coarser grids

Procedure:

- Several steps of a basic method on the full grid are performed first in order to smooth out the error
- Select a subset of the grid points, and the iterative method is used to solve the problem on this coarse grid
- The coarse grid solution is interpolated back to the original grid
- The original method applied again for a few iterations.
- The method of descending through a sequence of coarser grids and then ascending back to the full grid is known as a V-cycle. A W-cycle results from visiting the coarse grid twice, with some smoothing steps in between

3. Krylov subspace methods

Krylov subspace is defined by:

$$K_m(A, r_0) = span\left\{r_0, \mathbf{A}r^0, \mathbf{A}^2 r^o, ...\mathbf{A}^{m-1} r^0\right\}$$

$r^0 = F - \mathbf{A}\phi_0$ is the residual for arbitrary error

The approximate solution ϕ_m lies in the space $\phi^0 + K_m(\mathbf{A}, r^0)$

Example: conjugate gradient method, Lanczos method, $F - \mathbf{A}\phi_m \perp K_m(\mathbf{A}, r_0)$

Other methods: **GMRES, MINRES, ORTHODIR** etc.