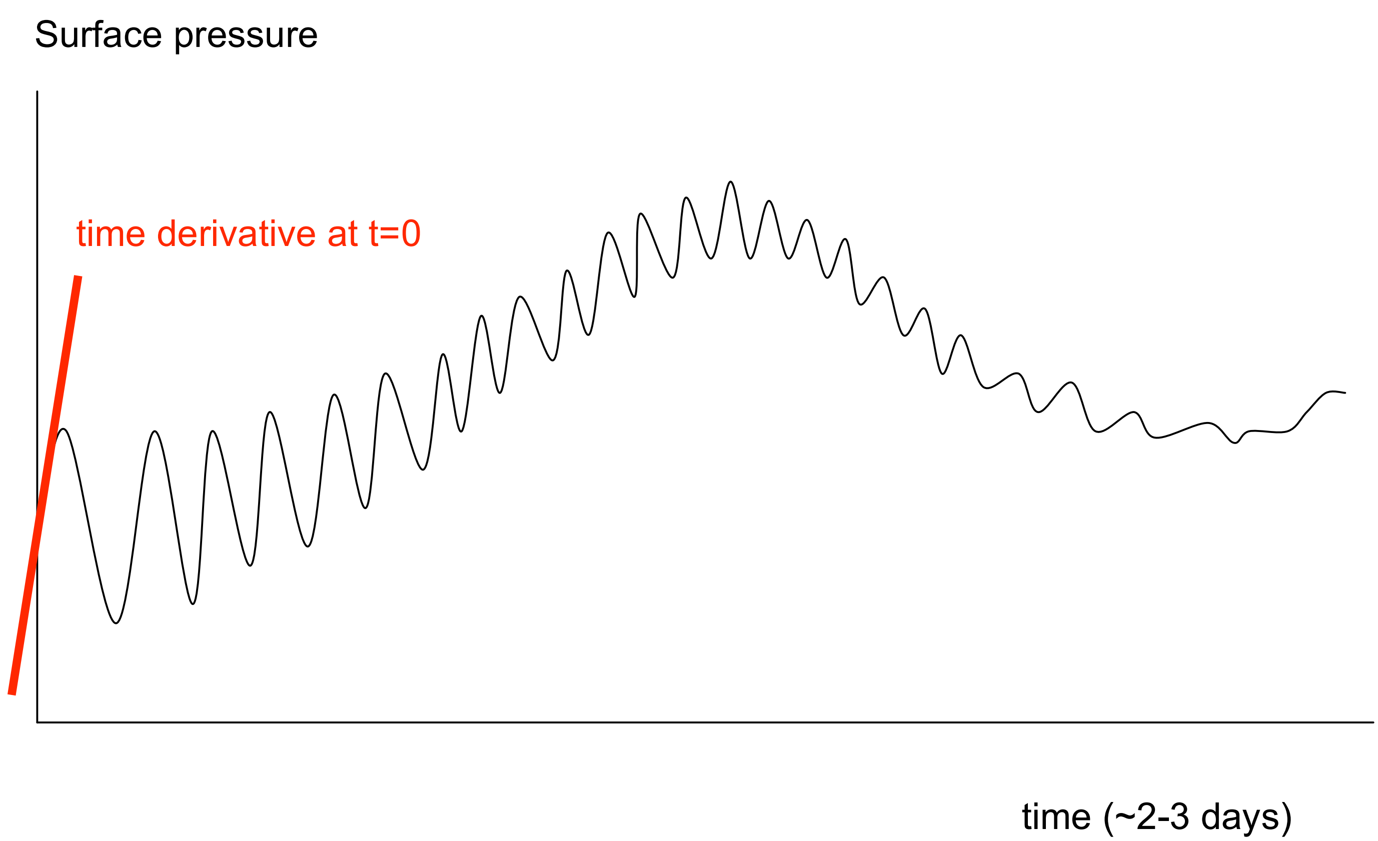
Chapter 2. The continuous equations

Fig. 1.2: Schematic of a forecast with slowly varying weather-related variations and superimposed high frequency Lamb waves. Note that even though the forecast of the slow waves is essentially unaffected by the presence of fast waves, the initial time derivative is much larger in magnitude, as obtained in Richardson (1922) experiment. Their amplitude decays because Lamb waves with rotation are dispersive



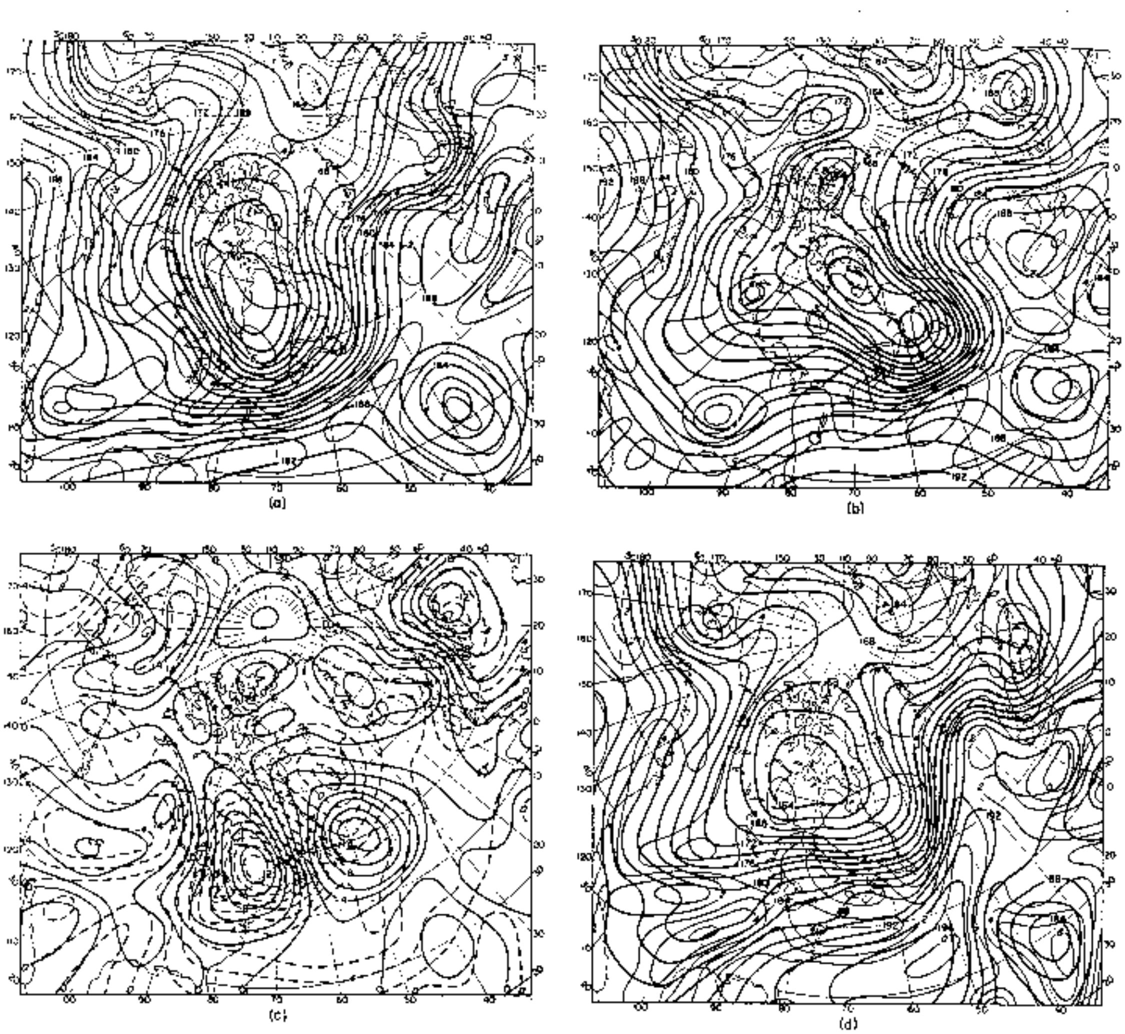


Fig 1.3 . Forecast of January 30, 1949,0300 GMT: (a) observed z (heavy lines) and $\zeta + f$ (light lines) at t=0; (b) observed z and $\zeta + f$ at t = 24 hr; (c) observed (continuous lines) and computed (broken lines) 24-hr height change; (d) computed z and $\zeta + f$ at t = 24 hr. The height unit is 100 ft and the unit of vorticity is 1/3 X 10^{-4} sec $^{-1}$. (Reproduced from the Compendium of Meteorology, with permission of the American Meteorological Society)

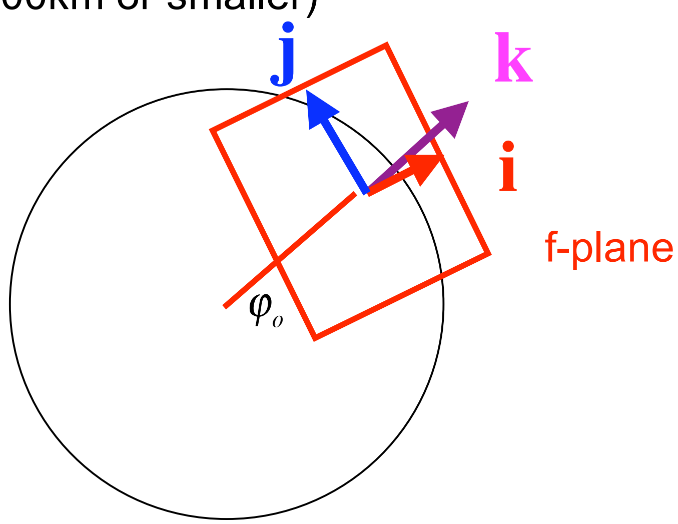
2.3 Basic wave oscillations in the atmosphere

In order to understand the problems in Richardson's result in 1922 (Fig. 1.2), and the effect of the filtering approximations introduced by Charney et al (1950), we need to have a basic understanding of the characteristics of the different types of waves present in the atmosphere.

The characteristics of these waves, (fast sound and gravity waves, and slow weather waves), have also profound implications for the present use of hydrostatic and non-hydrostatic models.

The three types of waves are present in the solutions of the governing equations, and different approximations such as the hydrostatic, the quasi-geostrophic and the anelastic approximations are designed to filter out some of them.

To simplify the analysis we make a tangent plane or "f-plane" approximation. We consider motions with horizontal scales L smaller than the radius of the earth. $L << a = 6400 km$ (e.g., *L*=1000km or smaller)



On this tangent plane we can approximate the spherical coordinates by

$$\frac{1}{a\cos\varphi_o}\frac{\partial}{\partial\lambda} \approx \frac{\partial}{\partial x}, \ \frac{1}{a}\frac{\partial}{\partial\varphi} \approx \frac{\partial}{\partial y}, \ f \approx 2\Omega\sin\varphi_o$$

and ignore the metric terms, since u/(a tanφ) is small compared to Ω.

The governing equations on an f-plane (rotating with the local vertical component of the Earth rotation) are

$$\frac{du}{dt} = +fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{a}$$

$$\frac{dv}{dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{b}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad \text{c}$$

$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \qquad \text{d}$$

$$\frac{ds}{dt} = \frac{Q}{T}; \ s = C_p \ln\theta \qquad \text{e}$$

$$p = \rho RT \qquad \text{f}$$

(3.1)

Simplify further:

- basic state at rest $u_0 = v_0 = w_0 = 0$.

From (3.1) a and b $\left(\frac{du}{dt} = +fv - \frac{1}{\rho}\frac{\partial p}{\partial x}\right)$,

p_0 does not depend on x, y, so that $p_0 = p_0(z)$

From (3.1)c, $\left(\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\right)$, ρ_0 (and all other basic state thermodynamic variables) also depend on z only.

- Assume that the motion is adiabatic and frictionless, $Q = 0, \mathbf{F} = 0$.

- Consider *small perturbations* $p = p_0 + p'$, etc. so that we can linearize the equations (neglect terms which are products of perturbations). For convenience, we define $u^* = \rho_0 u'; v^* = \rho_0 v'; w^* = \rho_0 w'; s^* = \rho_0 s'$.

- Since $s = C_p \ln\theta$ the basic state stratification is given by

$$\frac{ds_0}{dz} = C_p \frac{1}{\theta_0}\frac{d\theta_0}{dz}$$

- The perturbation equations are then

$$\frac{\partial u^*}{\partial t} = +fv^* - \frac{\partial p'}{\partial x} \qquad \text{a}$$

$$\frac{\partial v^*}{\partial t} = -fu^* - \frac{\partial p'}{\partial y} \qquad \text{b}$$

$$\frac{\partial w^*}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \qquad \text{c}$$

$$\frac{\partial \rho'}{\partial t} = -\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right) \qquad \text{d}$$

$$\frac{\partial s^*}{\partial t} = -w^* \frac{ds_0}{dz} \qquad \text{e} \qquad (3.2)$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} \qquad \text{f}$$

where

$$s^* = \rho_0 C_p \frac{\theta'}{\theta_0} = \rho_0 C_p \left(\frac{T'}{T_0} - \frac{R}{C_p}\frac{p'}{p_0}\right) = C_p \left(\frac{p'}{\gamma R T_0} - \rho'\right)$$

g

Exercise: Derive (3.2)a-g, recalling that

$$p = \rho RT \ , \theta = T\left(\frac{p_0}{p}\right)^{R/C_p} \ , \ C_p = R + C_v \ ,$$

$\gamma = C_p / C_v = 1.4$, and $c_s^2 = \gamma RT \approx (320 m / \sec)^2$ is the square of the speed of sound.

2.3.1 Pure types of plane wave solutions

We first consider *special cases with pure wave solutions*. (They exist in their pure form only under very simplified assumptions. However, if we understand their basic characteristics, we will understand their role in the full nonlinear models, and the methodology used for filtering some of the waves out).

We will be assuming plane wave solutions aligning the x-axis along the direction of propagation:

$$(u^*, v^*, w^*, p') = (U, V, W, P)e^{i(kx + mz - \nu t)} \qquad (3.3)$$

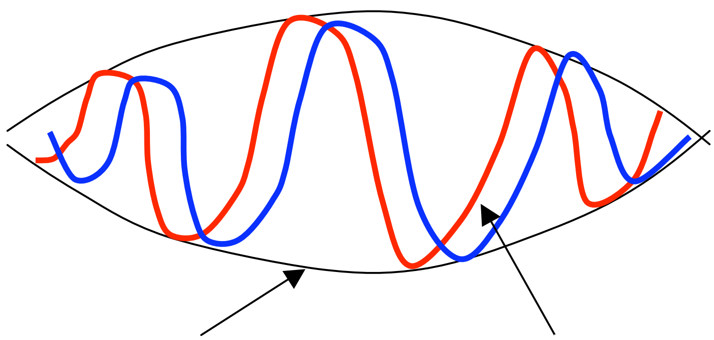
$k = 2\pi / L_x, m = 2\pi / L_z$ are horizontal and vertical wavenumbers,

$\nu = 2\pi / T$ is the frequency, and

U, V, W, P are constant amplitudes.

We will aim to derive the **frequency dispersion relationship (FDR)** $\nu = f(k, m, parameters)$ for each type of wave

The FDR gives us not only the frequency, but also the *phase speed* components ($\nu / k, \nu / m$) as well as the *group velocity* components ($\partial\nu / \partial k, \partial\nu / \partial m$). The phase speed is the speed of individual wave crests and valleys, and the group velocity is the speed at which wave energy propagates in the horizontal and vertical directions.



A pure type of wave occurs under idealized conditions such as no rotation, no stratification for sound waves, but its basic characteristics are retained even if the ideal conditions are not valid (sound waves are still present but slightly modified in the presence of rotation and stratification).

1) ***Pure** sound waves*:

In order to make things as simple as possible, we neglect rotation, stratification and gravity, since they don't affect sound waves much:

$$f = 0, \quad g = 0, \quad \frac{ds_0}{dz} = 0 \,.$$

From (3.2)e ($\frac{\partial s^*}{\partial t} = -w^* \frac{ds_0}{dz}$), we have $s^* = 0$ (recall that s^* is a perturbation, and if it was constant, we would have included its value into the basic state s_o).

Therefore from (3.2)g, ($s^* = C_p(\frac{p'}{\gamma R T_0} - \rho') = 0$) we get

$p' = c_s^2 \rho'$, and equations (3.2) reduce to

$$\frac{\partial u^*}{\partial t} = -\frac{\partial p'}{\partial x} \qquad \text{a}$$

$$\frac{\partial v^*}{\partial t} = -\frac{\partial p'}{\partial y} \qquad \text{b}$$

$$\frac{\partial w^*}{\partial t} = -\frac{\partial p'}{\partial z} \qquad \text{c} \quad (3.4)$$

$$\frac{1}{c_s^2}\frac{\partial p'}{\partial t} = -(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}) \qquad \text{d}$$

They show that sound waves occur through adiabatic expansion and contraction (3D divergence), and that the pressure perturbation is proportional to the density perturbation.

Assuming plane wave solutions $(u^*, v^*, w^*, p') = (U, V, W, P)e^{i(kx+mz-\nu t)}$ (3.3), with the x-axis along the horizontal direction of the waves, and replacing into (3.4) we get

$$
\begin{aligned}
-i\nu U &= -ikP && \text{a}\\
-i\nu V &= 0 && \text{b}\\
-i\nu W &= -imP && \text{c}\\
-i\nu P &= -c_s^2(ikU + imW) && \text{d}
\end{aligned}
\qquad (3.5)
$$

From (3.5)b V=0, and replacing U and W from a and c into d, we get the Frequency Dispersion Relationship (FDR):

$$\nu^2 = c_s^2(k^2 + m^2) \qquad (3.6)$$

These are sound waves that propagate through air compression or 3-dimensional divergence. The components of the phase velocity are $\left(\frac{\nu}{k}, \frac{\nu}{m}\right)$ and the total phase velocity is $\frac{\nu}{\sqrt{k^2 + m^2}} = \pm c_s$.

Repeat of (3.2) (for the next section, sound waves in the presence of stratification)

$$\frac{\partial u^*}{\partial t} = +fv^* - \frac{\partial p'}{\partial x} \qquad \text{a}$$

$$\frac{\partial v^*}{\partial t} = -fu^* - \frac{\partial p'}{\partial y} \qquad \text{b}$$

$$\frac{\partial w^*}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \qquad \text{c}$$

$$\frac{\partial \rho'}{\partial t} = -\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z}\right) \qquad \text{d}$$

$$\frac{\partial s^*}{\partial t} = -w^* \frac{ds_0}{dz} \qquad \text{e}$$

$$\frac{p'}{p_0} = \frac{\rho'}{\rho_0} + \frac{T'}{T_0} \qquad \text{f}$$

where

$$s^* = \rho_0 C_p \frac{\theta'}{\theta_0} = \rho_0 C_p \left(\frac{T'}{T_0} - \frac{R}{C_p}\frac{p'}{p_0}\right) = C_p \left(\frac{p'}{\gamma R T_0} - \rho'\right) \qquad \text{g}$$

$\frac{\partial s^*}{\partial t} = -w^* \frac{ds_0}{dz}$ is equivalent to $\frac{\partial \theta'}{\partial t} = -w^* \frac{d \ln \theta_0}{dz}$

2) *Lamb waves (horizontally propagating sound waves)*:

We now neglect rotation and assume that there is only horizontal propagation (no vertical velocity), but we allow for the fluid to be gravitationally stratified.

With f=0 and w*=0, we again have s*=0, and from (3.2)e ($\frac{\partial s^*}{\partial t} = -w^* \frac{ds_0}{dz}$) and (3.2)g $p' = c_s^2 \rho'$

From (3.2)c ($-\frac{\partial p'}{\partial z} - \rho' g = \frac{\partial w^*}{\partial t} = 0$)

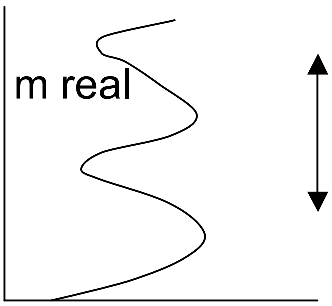
the flow is now hydrostatic: $\frac{\partial p'}{\partial z} = -\rho' g$.

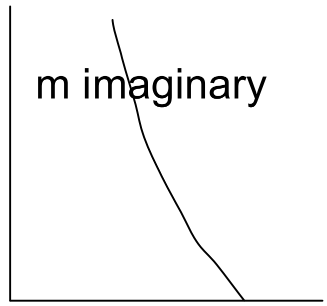
If we insert the same type of plane wave solutions

$$(u^*, v^*, w^*, p') = (U, V, W, P) e^{i(kx + mz - \nu t)}$$

into (3.2), we find that $p' = P e^{-\frac{g}{c_s^2} z} e^{i(kx - \nu t)}$, i.e., the vertical wave number is imaginary $m = ig / c_s^2$, and the phase speed is $\nu^2/k^2 = c_s^2$.

Since the vertical wavenumber is imaginary, there is no vertical propagation, and the waves are *external*.



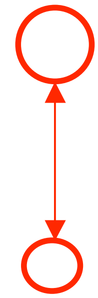


Therefore, the **Lamb waves** are a type of external horizontal sound wave, which is present in the solutions of models even when the hydrostatic approximation is made.

This is very important because it means primitive equation models (which make the hydrostatic approximation) contain these fast moving horizontal sound waves. We will see that the Lamb waves are also equivalent to the gravity waves in a shallow water model. Note also that the FDR is such that $\frac{\nu}{k} = \pm c_s$, so that the phase speed does not depend on the wave number.

This implies that the group velocity $\frac{\partial \nu}{\partial k} = \pm c_s$, it is also independent of the wave number, and as a result Lamb waves without rotation are non-dispersive, so that a package of waves will move together and not disperse. With rotation, they do disperse (different wave components have different group velocities).

2) ***Pure*** *vertical gravitational oscillations*



(Assume a parcel can only move up or down)

Now we neglect rotation and pressure perturbations, $f = p' = 0$, so that there is no horizontal motion, but allow for vertical stratification. Equations (3.2) become

$$\frac{\partial w^*}{\partial t} = -\rho' g \qquad \text{a}$$

$$\frac{\partial \rho'}{\partial t} = \frac{w^*}{C_p}\frac{ds_0}{dz} = w^* \frac{d \ln \theta_0}{dz} \qquad \text{b} \qquad (3.7)$$

From these two equations we get

$$\frac{\partial^2 w^*}{\partial t^2} + N^2 w^* = 0 \qquad (3.8)$$

and from the continuity equation

$$\frac{\partial \rho'}{\partial t} = -\frac{\partial w^*}{\partial z} \qquad (3.9)$$

Replacing the plane wave solution (3.3) into (3.8) we obtain $\nu^2 = N^2$, where

$N^2 = g\dfrac{d\ln\theta_0}{dz}$ is the square of the *Brunt-Vaïsälä frequency*.

$$N^2 = \frac{g d\theta_0}{\theta_0 dz} \sim \frac{g\ \Delta\theta_0}{\theta_0\ \Delta z} \sim \frac{10m/s^2 \bullet 30K}{300K \bullet 10km} \sim 10^{-4}\sec^{-2}$$

A typical value for the atmosphere is $N \sim 10^{-2}\sec^{-1}$. A parcel displaced in a stable atmosphere, will oscillate vertically with frequency *N*. Equations (3.7)b and (3.9) show that the amplitude of *w** will decrease with height as $e^{-(d\ln\theta_0/dz)z}$.

4) ***Pure*** *Inertia oscillations*

Inertia oscillations are horizontal and are due to the basic rotation.

We now assume that $p' = 0, \dfrac{ds_0}{dz} = 0$, no pressure perturbations and no stratification.

Then $s^* = 0$, and therefore, $\rho' = 0$.

Then the horizontal equations of motion become

$$\frac{\partial \mathbf{v}^*}{\partial t} = -f\mathbf{k}\times\mathbf{v}^*$$ and therefore

$$\frac{\partial^2 \mathbf{v}^*}{\partial t^2} = f\mathbf{k} \times (f\mathbf{k} \times \mathbf{v}^*) = -f^2 \mathbf{v}^* \qquad (3.10)$$

As indicated by (3.10), the frequency of inertia oscillations is $\nu = \pm f$, with the acceleration perpendicular to the wind, corresponding to a circular wind oscillation. In the presence of a basic flow, there is also a translation, and the trajectories look like Fig. 2.3.1. They are seen in the track of ocean buoys.

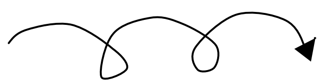


Fig. 2.3.1: Schematic of an inertial oscillation in the NH in the presence of a basic flow to the right.

5) ***Lamb waves in the presence of rotation* and *geostrophic modes***

(these are the big problem in primitive equations models as shown by Richardson's results):

We now consider the same case as 2) of horizontally propagating Lamb waves, but without neglecting rotation, i.e., $f \neq 0$

The vertical velocity is still zero. From $w^* = 0$ and (3.2)c we have again $p' = c_s^2 \rho'$, and the hydrostatic balance in (3.2) g implies then $\frac{\partial p'}{\partial z} = -\frac{g}{c_s^2} p' = -\frac{p'}{\gamma H}$.

Therefore the 3-dimensional perturbations can be written as

$$p'(x, y, z, t) = p'(x, y, o, t) e^{-z/\gamma H},$$

where $\gamma H = c_s^2 / g$.

The system of equations (3.2) becomes

$$\frac{\partial \mathbf{v}^*}{\partial t} = -f\mathbf{k} \times \mathbf{v}^* - \nabla p'$$

$$\frac{\partial p'}{\partial t} = -c_s^2 \nabla . \mathbf{v}^* \qquad (3.11)$$

Note: This system is completely analogous to the linearized *shallow water equations* (SWE) which are widely used in NWP as the simplest primitive equations model:

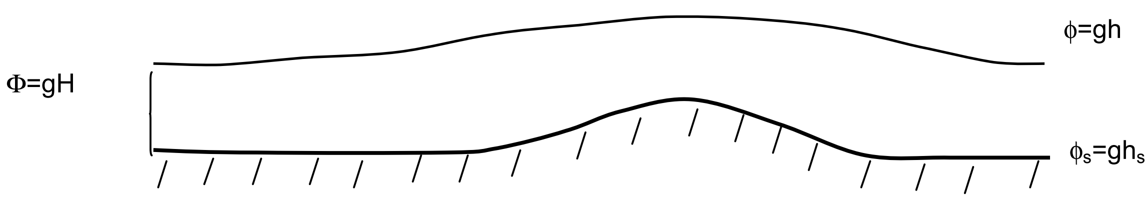
$$\frac{\partial \mathbf{v}^*}{\partial t} = -f\mathbf{k} \times \mathbf{v}^* - \nabla \phi'$$

$$\frac{\partial \phi'}{\partial t} = -\Phi \nabla . \mathbf{v}^* \quad \text{where}$$

(3.12)

$$\phi = \Phi + \phi'$$

Fig. 2.5: Schematic of the shallow water model, a hydrostatic, incompressible fluid with a rigid bottom $h_s(x,y)$, a free surface $h(x,y,t)$, and horizontal scales L much larger than the mean vertical scale H.



If we assume plane wave solutions of the form $(u^*, v^*, p') = (U, V, P)e^{-i(kx - \nu t)}$, and replace in (3.12) we obtain:

$$-i\nu U = fV - ikP$$

$$-i\nu V = -fU \qquad (3.13)$$

$$-i\nu P = -c_s^2 ikU$$

Therefore the FDR is

$$\nu(\nu^2 - f^2 - c_s^2 k^2) = 0 \quad (3.14)$$

Note that this FDR contains *two* types of solutions:

one type is $\nu^2 = f^2 + c_s^2 k^2$, Lamb waves (i.e., horizontal sound waves modified by inertia, that is rotation), or inertia-Lamb waves.

In the SWE analogue, these are inertia-gravity waves (external gravity waves modified by inertia), $\nu^2 = f^2 + \Phi k^2$

Note that in the presence of rotation the phase speed and group velocity depend on the wave number: rotation makes Lamb waves dispersive (and this helps with the problem of getting rid of noise in the initial conditions as in Fig. 1.2).

In (3.14) the *second type of solution* (and for us the most important!) is *the* ***steady state solution*** ν=0.

This means that $\frac{\partial()}{\partial t} = -i\nu() = 0$ for all variables.

Without the presence of rotation, this steady state solution would be trivial: $u^* = v^* = w^* = p' = 0$.

But *with rotation*, an examination of (3.13) or (3.11)

$$\frac{\partial \mathbf{v}^*}{\partial t} = -f\mathbf{k}\times\mathbf{v}^* - \nabla p'$$

$$\frac{\partial p'}{\partial t} = -c_s^2 \nabla.\mathbf{v}^*$$

shows that this is *the geostrophic mode*:

$$U = 0,\ \nabla.\mathbf{v}^* = \partial u^*/\partial x = 0,$$

<u>but</u> $V = ikP/f$, i.e.,

$$v^* = \frac{1}{f}\frac{\partial p'}{\partial x}.$$

This is a steady state, but nontrivial, geostrophic solution. If we add a dependence of *f* on latitude, the geostrophic solution becomes the Rossby waves solution, which is not steady state, but is still much *slower* than gravity waves or sound waves.

2.3.2 General wave solution of the perturbation equations in a resting, isothermal atmosphere

So far we have been making drastic approximations to obtain "pure" elementary waves (sound, inertia and gravity oscillations).

We now consider a more general case, including all waves simultaneously. We consider again the equations for small perturbations (3.2), and assume a resting, isothermal basic state in the atmosphere: $T_0(z) = T_{00}$, a constant. Then

$$N^2 = g\frac{d\ln\theta}{dz} = -g\kappa\frac{d\ln p_0}{dz} \tag{3.15}$$

where $\kappa = R/C_p = 0.4$.

Since the basic state is hydrostatic,

$$N^2 = g\kappa\frac{\rho_0 g}{p_0} = g\kappa\frac{g}{RT_0} = \frac{g\kappa}{H} \tag{3.16}$$

For an isothermal atmosphere, both N^2 and the scale height $H = RT/g$ are constant.

We continue considering an f-plane, a reasonable approximation for horizontal scales L small compared to the radius of the earth: $L << a$.

If *L* were not small compared with the radius of the earth, we would have to take into account the variation of the Coriolis parameter with latitude, and spherical geometry.

With some manipulation, assuming that the waves propagate along the x-axis, and there is no y-dependence, the perturbation equations (3.2) become

$$\frac{\partial u^*}{\partial t} = +fv^* - \frac{\partial p'}{\partial x} \qquad \text{a}$$

$$\frac{\partial v^*}{\partial t} = -fu^* \qquad \text{b}$$

$$\alpha\frac{\partial w^*}{\partial t} = -\frac{\partial p'}{\partial z} - \rho' g \qquad \text{c}$$

$$\beta\frac{\partial \rho'}{\partial t} = -\left(\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z}\right) \qquad \text{d}$$

$$\frac{g}{C_p}\frac{\partial s^*}{\partial t} = -w^* N^2 \qquad \text{e} \qquad (3.17)$$

$$s^* = C_p\left(\frac{p'}{c_s^2} - \rho'\right) \qquad \text{f}$$

In these equations we have introduced two constants α and β as *markers* for the hydrostatic and the quasi-Boussinesq approximations respectively. They can take the value 1 or 0.

If we make $\alpha = 0$, it indicates that we are making the *hydrostatic* approximation, i.e., neglecting the vertical acceleration in (3.17)c.

If we make $\beta = 0$ it indicates that we are making the *anelastic or quasi-Boussinesq* approximation, i.e., assuming that the mass weighted 3-dimensional divergence is zero. Otherwise the markers take the value 1. These markers will be used in the next section, where we discuss filtering approximations.

We now try plane wave solutions where the basic state is a function of z, of the form

$$(u^*, v^*, w^*, p', \rho') = (U(z), V(z), W(z), P(z), R(z))e^{i(kx-\nu t)}$$

(3.18)

Instead of assuming a z-dependence of the form $e^{i(mz)}$, we will determine it explicitly. If the horizontal scale is not small compared to the radius of the earth, L~a, then the solutions are of the form

$$(u^*, v^*, w^*, p', \rho') = (U(z), V(z), W(z), P(z), R(z))A(\varphi)e^{i(s\lambda-\nu t)}$$

The equation obtained for A(φ) is *the Laplace-tidal equation*.

Replacing the assumed form of the solution (3.18) into (3.17) we get

$$-i\nu U = -ikP + fV \qquad \text{a}$$

$$-i\nu V = -fU \qquad \text{b}$$

$$-i\nu\alpha W = -Rg - \frac{dP}{dz} \qquad \text{c}$$

$$-i\nu\beta R = -ikU - \frac{dW}{dz} \qquad \text{d} \quad (3.19)$$

$$-i\nu\left(\frac{P}{c_s^2} - R\right) = -W\frac{N^2}{g} \qquad \text{e}$$

From a ($-i\nu U = -ikP + fV$) and b ($-i\nu V = -fU$)

$$U = \frac{k\nu}{\nu^2 - f^2}P \qquad \text{f}$$

From d ($-i\nu\beta R = -ikU - \frac{dW}{dz}$) and f we get:

$$\beta R = \frac{k^2}{\nu^2 - f^2}P - \frac{i}{\nu}\frac{dW}{dz} \qquad \text{g}$$

From c ($-i\nu\alpha W = -Rg - \frac{dP}{dz}$) and e

($-i\nu(\frac{P}{c_s^2} - R) = -W\frac{N^2}{g}$) we get

$$\frac{dP}{dz} + \frac{g}{c_s^2}P = \frac{i}{\nu}(\nu^2\alpha - N^2)W \qquad \text{h}$$

From e and g

$$\frac{dW}{dz} + \beta\frac{N^2}{g}W = \frac{i\nu}{c_s^2}[\frac{\beta(\nu^2 - f^2) - c_s^2 k^2}{\nu^2 - f^2}]P \qquad \text{i}$$

From h and i

$$(\frac{d}{dz} + \frac{g}{c_s^2})(\frac{d}{dz} + \beta\frac{N^2}{g})W =$$
$$= -\frac{1}{c_s^2}[\frac{(\beta(\nu^2 - f^2) - c_s^2 k^2)(\nu^2\alpha - N^2)}{\nu^2 - f^2}]W \qquad (3.20)$$

or a similar equation for P.

This last equation

$$(\frac{d}{dz}+\frac{g}{c_s^2})(\frac{d}{dz}+\beta\frac{N^2}{g})W=-\frac{1}{c_s^2}[\frac{(\beta(\nu^2-f^2)-c_s^2k^2)(\nu^2\alpha-N^2)}{\nu^2-f^2}]W$$

is of the form $\frac{d^2W}{dz^2}+A\frac{dW}{dz}+BW=0.$

In order to eliminate the first derivative, we try a substitution of the form $W=e^{\delta z}\Omega$, and obtain $\frac{d^2\Omega}{dz^2}+C\Omega=0.$

This requires that we choose $\delta=-A/2$, and in that case $C=B-A^2/4$.

From (3.20), the variable substitution and additional sweat, we finally obtain

$$\frac{d^2\Omega}{dz^2}+n^2\Omega=0 \qquad (3.21)$$

where

$$n^2=\frac{(\beta(\nu^2-f^2)-c_s^2k^2)(\nu^2\alpha-N^2)}{c_s^2(\nu^2-f^2)}-\frac{1}{4}(\beta\frac{N^2}{g}+\frac{g}{c_s^2})^2 \qquad (3.22)$$

This is the frequency dispersion relationship for waves in an atmosphere with an isothermal basic state. Given a horizontal structure of the wave (k), and its frequency (ν), (3.22) determines the vertical structure (n) of Ω (and W), and viceversa. The same FDR would have been obtained making the substitution $Q=e^{-\delta z}P$, and solving for Q.

Equation (3.22) indicates that depending on the sign of n^2 we can have either external or internal wave solutions:

a) External waves:

If $n^2 < 0$, the vertical wave number n is imaginary, $n = im$.

The solution of (3.21) is then $\Omega = Ae^{mz} + Be^{-mz}$, or, going back to the vertical velocity,

$$w^*(x,z,t) = e^{i(kx-\nu t)} e^{-\frac{1}{2}[\beta\frac{N^2}{g}+\frac{g}{c_s^2}]z} (Ae^{mz} + Be^{-mz}) \quad (3.23)$$

These are external waves (the waves do not oscillate in the vertical, and therefore do not propagate vertically).

If the boundary condition at the ground is that the vertical velocity is zero, then $\Omega = Ae^{mz} + Be^{-mz} = 0$ at z=0, so that A+B=0, and

$$w^*(x,z,t) = e^{i(kx-\nu t)} e^{-\frac{1}{2}[\beta\frac{N^2}{g}+\frac{g}{c_s^2}]z} 2A\sinh(mz),$$

which has an exponential behavior in z. Since $\sinh(mz)$ cannot be zero above the ground, a top boundary condition of a rigid top can only be satisfied if A=0.

In other words, we cannot have external waves with rigid top and bottom boundary conditions: external waves require a free surface at the top (or at the bottom).

b) Internal waves:

If $n^2 > 0$, the vertical wave number n is real:

$$w^*(x,z,t) = e^{i(kx-\nu t)}(Ae^{inz} + Be^{-inz})e^{-\frac{1}{2}[\beta\frac{N^2}{g}+\frac{g}{c_s^2}]z} \quad (3.24)$$

A, B are determined from the boundary conditions. Now there is both vertical and horizontal propagation. For example, if there is a rigid bottom, we have again *A+B=0*, and the solution becomes

$$w^*(x,z,t) = A[e^{i(kx+nz-\nu t)} - e^{i(kx-nz-\nu t)}]e^{-\frac{1}{2}[\beta\frac{N^2}{g}+\frac{g}{c_s^2}]z}$$

The shape of internal waves in the vertical is shown schematically in Fig. 2.3.2

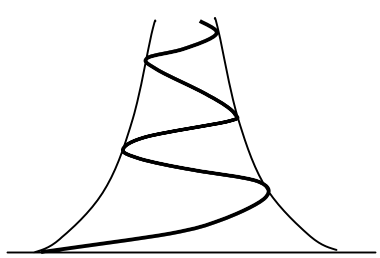


Fig. 2.3.2: Schematic of internal (vertically propagating) waves.

2.3.3 Analysis of the FDR of wave solutions in a resting, isothermal atmosphere

We will now plot the general FDR equation (3.22). We assume $T_{00} = 250K$ and $f = 2\Omega \sin 45^0 \approx 10^{-4} \sec^{-1}$.

Then, the speed of sound is $c_s^2 = \gamma RT \approx 10^5 m^2 / \sec^2$, or $c_s \approx 320 \text{m} / \sec$,

the scale height is $H = RT / g = 7.3\text{km=7300m}$,

the Brunt-Vaïsälä frequency is

$$N^2 = g \frac{d \ln \theta_0}{dz} = \frac{g\kappa}{H} \approx 4 * 10^{-4} \sec^{-2}; \qquad N \approx 2 * 10^{-2} \sec^{-1}$$

with a typical period for vertical oscillations of $2\pi / N \approx 300 \sec = 5 \min$.

Note that the frequency associated with inertial oscillations is much lower than the frequency associated with gravitational oscillations.

$$f \sim 10^{-4} \sec^{-1} << N \sim 10^{-2} \sec^{-1} \qquad (3.25)$$

We first plot in Fig. 2.3.3 the FDR (3.22), with α=β=1, i.e., without making either the hydrostatic or the quasi-Boussinesq approximations. Note that this equation contains 4 solutions for the frequency ν, plus an additional solution

$\nu = 0$, the geostrophic mode (note that it satisfies nontrivially eq (3.19)).

Repeat

$$n^2 = \frac{(\beta(\nu^2 - f^2) - c_s^2 k^2)(\nu^2\alpha - N^2)}{c_s^2(\nu^2 - f^2)} - \frac{1}{4}(\beta\frac{N^2}{g} + \frac{g}{c_s^2})^2$$

Fig. 2.3.3: Schematic of the frequencies of small perturbations in an isothermal resting atmosphere as a function of k, the horizontal wave number (the horizontal scale is its inverse), and the vertical wave number n. Shaded regions represent $n^2<0$, external waves.