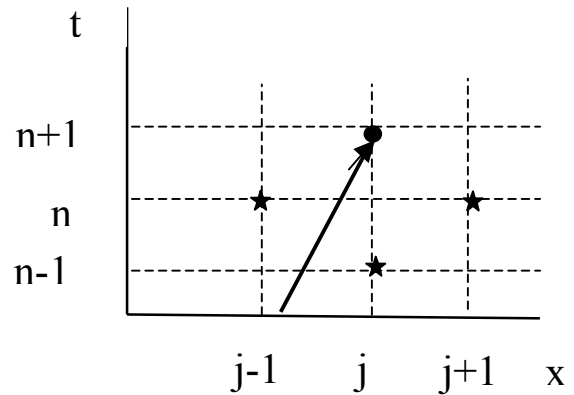**<u>Example 2: Leap-frog scheme for the wave equation</u>**.

PDE: $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$

FDE: $\dfrac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + c\dfrac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0$

This is the most popular of all schemes used for hyperbolic equations: centered in space, centered in time. It requires one computation of the time derivative per time step (compared to 4 computations for Runge Kutta).

It is easy to see that the Leap-frog scheme is consistent, and the local truncation error is of second order in space and time (exercise).

Stability: We use the von Neumann criterion, with an amplification factor ρ_p^n:

Assume $U_j^n = \sum_p Z_p^n e^{ipj} = \sum_p A\rho_p^n e^{ipj}$

Replace in the FDE

$$\dfrac{\rho_p^{n+1} - \rho_p^{n-1}}{2\Delta t} + c\dfrac{\rho_p^n(e^{ip} - e^{-ip})}{2\Delta x} = 0$$

from which we get a quadratic equation for ρ_p

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$$\rho_p{}^2 + 2i\mu \sin p \rho_p - 1 = 0$$

Because we have three, not two, time levels ρ^{n+1}, ρ^n and ρ^{n-1} we have two solutions for the amplification factor ρ:

$$\rho_p = (-i\mu \sin p) \pm \sqrt{(-\mu^2 \sin^2 p + 1)}$$

Since the last term in the quadratic equation is −1, and this is the product of the roots, the term inside the root $(-\mu^2 \sin^2 p + 1)$ must be real, since otherwise the roots would be purely imaginary, and one of them would be larger than one, which violates the stability criterion. In order for $\sqrt{(-\mu^2 (\sin p)^2 + 1)}$ to be real for all p, we must have $\mu^2 \leq 1$. The stability condition for the leap-frog scheme becomes

$$-1 \leq c\Delta t / \Delta x \leq 1$$

Question: What is the most unstable $p = \dfrac{2\pi\Delta x}{L}$? Which wavelength will start blowing up faster?

We can actually find the exact solution of the Leap Frog FDE as well as of the PDE. Recall that the PDE $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$ has plane wave solutions of the form

$$u = \sum_k A_k e^{ik(x-ct)} = \sum_k A_k e^{i(kx-\omega t)}$$, since the exact solution
is of the form u(x,t)=u(x-ct,0).

The Frequency Dispersion Relationship $\omega = kc$ gives the
exact frequency of the PDE for every wavenumber k.

By analogy we try to find solutions of the FDE of the form
$A_p e^{i(pj-\theta n)}$, where θ=νΔt represents the computational
frequency v multiplied by Δt (the computational frequency
V is in general different than the exact frequency ω).

Replacing $A_p e^{i(pj-\theta n)}$ in the FDE $\dfrac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + c\dfrac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0$

and dividing by $e^{i(pj-\theta n)}$, we get

$(e^{-i\theta} - e^{i\theta}) + \mu(e^{ip} - e^{-ip}) = 0$ or

$\sin\theta = \mu\sin p$ the FDR for the Leap-Frog scheme.

Because $\sin\theta = \sin(\pi - \theta)$, the two solutions for the finite
difference FDR are

$\theta_1 = \arcsin(\mu\sin p)$
$\theta_2 = \pi - \arcsin(\mu\sin p)$

Replacing into the FDR, and assuming that the initial
amplitude for the wavenumber p is 1, we obtain that the

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solution of the FDE is a sum of two terms corresponding to θ_1 and θ_2 respectively:

$$U_j^n = A_p e^{i(pj-\theta n)} + (1-A_p)e^{i(pj+\theta n)}(-1)^n$$

where $\theta = \arcsin(\mu \sin p)$, and $e^{i\pi} = 1$

(Shorter derivation: when we assume solutions of the form $\rho_p e^{i(pj-\theta n)}$, and plug into FDE. The amplification factor is

$$\rho = e^{-i\theta} = \cos\theta - i\sin\theta = -i\mu \sin p \pm \sqrt{1-\mu^2 \sin^2 p}$$, i.e.,

$\sin\theta = \mu \sin p$, with two solutions as indicated above).

Of the two terms in the solution,

$$U_j^n = A_p e^{i(pj-\theta n)} + (1-A_p)e^{i(pj+\theta n)}(-1)^n$$

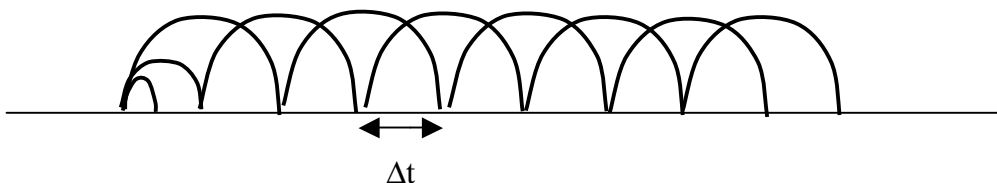the first one is the "legitimate" solution, which approximates the PDE solution.

Note that the second term changes sign every time step, and it moves in the wrong direction: this unphysical term is called "computational mode". It arises because the leapfrog scheme has three time levels, rather than two, giving rise to an additional spurious solution.

Although the LF scheme is simple and accurate, its 3-time level character gives rise to two problems that need to be dealt with:

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The first is that the LF needs a special initial step to get to the first time level U^1 from the initial conditions U^0, before the LF can be started (schematic Fig. 3.5). This could be done in several simple ways:

a) Simply set U^1= U^0. Since $u^1 = u^0 + u_t \Delta t + ...$, this introduces errors of order O(Δt), and is not recommended.

b) Use for the first time step a forward time scheme (exercise 5: show that this scheme is unstable for hyperbolic equations). The forward scheme has truncation errors of order O(Δt), but since the time step is only used once, its contribution to the global error gets multiplied by Δt, so that the total error is still of O(Δt)^2. For the same reason, the computational instability is not a significant problem. Alternatively, use an Euler-backwards (Matsuno) scheme for the first time step (see Table 1).

c) Use half (or a quarter, eighth, etc.) of the initial time step for the forward time step (Fig. 3.5), followed by LF time steps. This will halve (or reduce by a quarter, eighth, etc.) the error introduced in the unstable first step.

Fig. 3.5 : Schematic of the Leap-Frog scheme with a half time step starting step.

Δt

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For nonlinear problems, the leap-frog scheme has a tendency to increase the amplitude of the computational mode with time, separating the space dependence in a checkerboard fashion between the even and odd time steps. This can be solved by restarting every 50 steps or so, or by applying a Robert-Asselin time filter.

Robert-Asselin time filter (Robert, 1969, Asselin, 1972):

In order to get rid of the 2Δt computational mode, we would like to apply a weak time smoother to the solution

$$\overline{U}^n = U^n + \alpha(U^{n+1} - 2U^n + U^{n-1})$$

This smoother would reduce the amplitude of different frequencies ν by a factor $(1 - 4\alpha \sin^2 \frac{\nu \Delta t}{2})$. The computational mode, whose period is 2Δt would be reduced by (1-4α) every time step.

However, this smoother would require to keep in memory three copies of the field, U^{n+1}, U^n, U^{n-1} in order to compute $\overline{U}^n$. Robert proposed a clever solution:

After the Leap-Frog scheme is used to obtain the solution at t=(n+1)Δt, the time filter is slightly modified:

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$$\overline{U}^n = U^n + \alpha(U^{n+1} - 2U^n + \overline{U}^{n-1})$$

and the smoothed solution $\overline{U}^n$ replaces the solution at time n, so that only two copies of the current field have to be saved in memory, like in the original Leap-Frog.

Because the field at t=(n-1)Δt is replaced by the already filtered value, the filter introduces a slight distortion of the centered filter (Asselin, 1977). This filter is widely used with the Leap-Frog scheme, with α of the order of 1%.

Example 3: $\dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2} + bu$

This is the heat or diffusion equation with a "source of growth" bu.

FDE: $\dfrac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \dfrac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} + bU_j^n$

Exercise 8: Show that the amplification factor is

$$\rho = 1 - \frac{4\sigma\Delta t}{\Delta x^2} + b\Delta t \le 1 + O(\Delta t)$$

Therefore the stability criterion is still $\dfrac{\sigma\Delta t}{\Delta x^2} \le 1/2$, as we obtained with the criterion of the maximum.

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Exercise 9: Explain physically why the term $b\Delta t$ does not influence the stability criterion.

Example 4: An implicit scheme

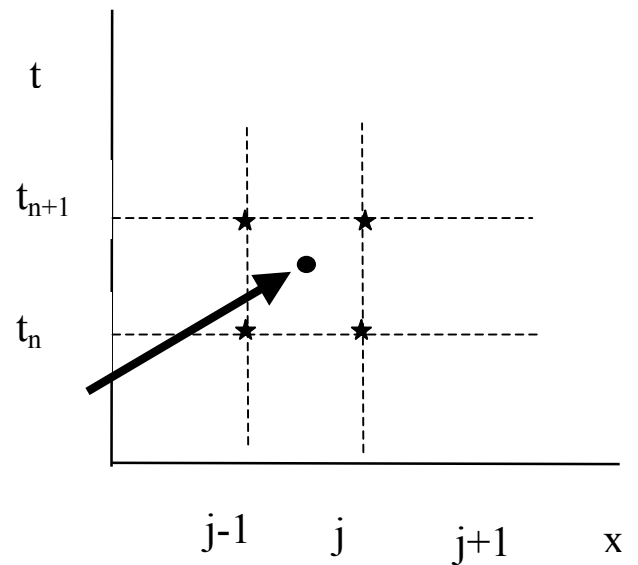PDE: $\dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0$
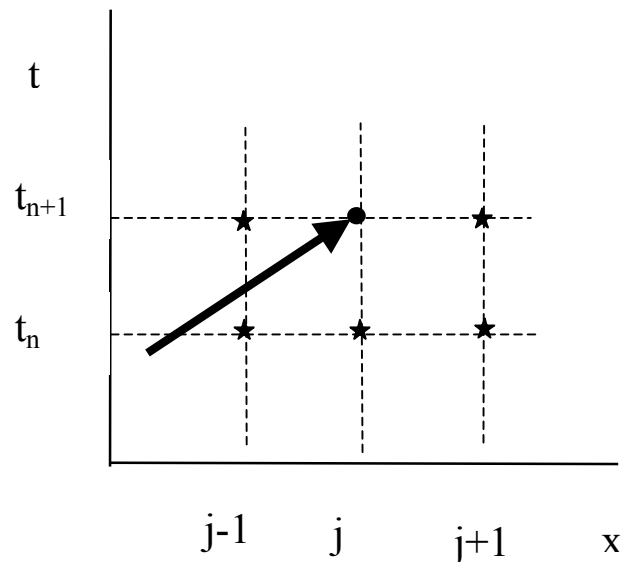
FDE:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + c\,\frac{(U_{j+1}^n - U_{j-1}^n) + (U_{j+1}^{n+1} - U_{j-1}^{n+1})}{2\Delta x} = 0$$

No extrapolation even for very fast c

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Another implicit FDE

$$\frac{U_j^{n+1} - U_j^n + U_{j+1}^{n+1} - U_{j+1}^n}{2\Delta t} + c\,\frac{\alpha(U_{j+1}^n - U_j^n) + (1-\alpha)(U_{j+1}^{n+1} - U_j^{n+1})}{\Delta x} = 0$$

The factor α determines the weight of the "old" time values compared to the "new" time values in the RHS of the FDE.

Using the von Neumann method, we replace
$$U_j^n = A\rho^n e^{ipj} = Ae^{i(pj-\theta n)}.$$

Note that the scheme is centered in time (if α=1/2) at the point $U_{j-1/2}^{n+1/2}$. For this reason, we multiply by $e^{-ip/2}$, and obtain the amplification factor:

$$\rho = \frac{\cos\dfrac{p}{2} - i2\mu\alpha\sin\dfrac{p}{2}}{\cos\dfrac{p}{2} + i2\mu(1-\alpha)\sin\dfrac{p}{2}}$$

or $\quad |\rho|^2 = \dfrac{1 + 4\mu^2\alpha^2\tan^2\dfrac{p}{2}}{1 + 4\mu^2(1-\alpha)^2\tan^2\dfrac{p}{2}}$

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This implies that $\rho \leq 1$ if $\alpha \leq 0.5$, i.e., if the new values are given at least as much weight as the old values in computing the RHS. In this case **there is no restriction on the size that Δt can take!** This result (absolute stability, independent of the Courant number) is typical of implicit time schemes.

In the schematic Fig. 3.6 we show that in an implicit scheme, a point at the new time level is influenced by all the values at the new level, which avoids extrapolation, and therefore is **absolutely stable**.

Note also that if $\alpha < 0.5$ (more weight is given to the new values than the old ones) the implicit time scheme reduces the amplitude of the solution: it is an example of a **damping scheme**. This property is useful to solve some problems such as spuriously growing mountain waves in semi-Lagrangian schemes.

Fig. 3.6: Schematic of an implicit scheme. The dot represents the value being updated and the stars the values that influence it. Note that with the implicit scheme there is no extrapolation, and therefore no limit to the size of Δt.
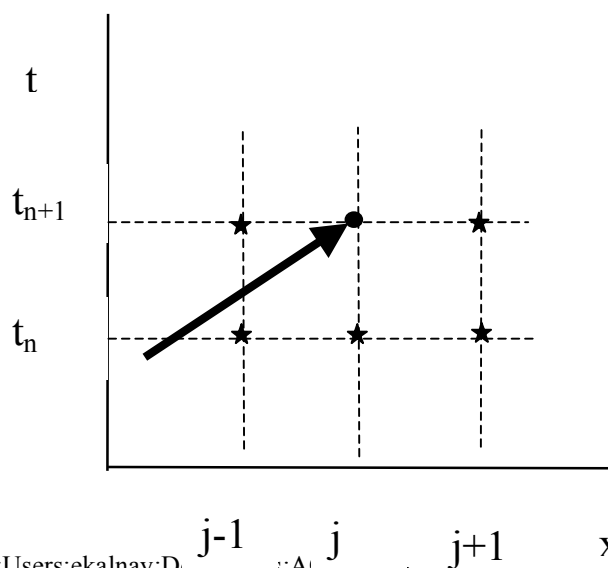
In summary, if we consider a marching equation

$$\frac{dU}{dt} = F(U)$$

explicit methods such as the forward scheme

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^n)$$

or the leap-frog scheme

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} = F(U^n)$$

are either **conditionally stable** (when there is a condition on the Courant number or the equivalent stability number for parabolic equations) or **absolutely unstable**.

A fully implicit scheme

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+1})$$

and a centered implicit scheme (Crank-Nicholson)

$$\frac{U^{n+1} - U^n}{\Delta t} = F(\frac{U^n + U^{n+1}}{2})$$

are **absolutely stable.** The latter is centered in space and in time, so it also has second order truncation errors, so it is quite accurate, and it only has two time levels so it does not have a computational mode, so it is an attractive scheme.

But, like all implicit schemes, it also has a great disadvantage. Since U^{n+1} appears on the left and on the right hand side, the solution for U^{n+1}, unlike explicit schemes, requires in general solving a system of equations.

If it involves only tridiagonal systems, this is not an obstacle, because there are fast methods to solve them. There are also methods, such as fractional steps (with each spatial direction solved successively), where one space dimension is considered at a time, that allow taking advantage of the large time steps allowed by implicit schemes without paying a large additional computational cost. (See scheme k in Table)

Moreover, we will see in the next section that the possibility of using a time step with a Courant number much larger than 1 in an implicit scheme does not imply that we will obtain accurate results economically. The implicit scheme maintains stability by **slowing down** the solutions, so that the slower waves do satisfy the CFL condition. For this reason implicit schemes are only useful for those modes (such as the Lamb wave or vertical sound waves) that are very fast but of little meteorological importance (semi-implicit schemes, next section).

Table 3.1 Time schemes for initial value problems dU/dt=F(U) (schemes a-i); dU/dt=F_1(U) +F_2(U) (schemes j-k).

a) $\dfrac{U^{n+1} - U^{n-1}}{2\Delta t} = F(U^n)$ Leap-Frog (good for hyperbolic

equations, unstable for parabolic equations)

a') $\dfrac{U^{n+1} - \overline{U}^{n-1}}{2\Delta t} = F(U^n)$; $\overline{U}^n = U^n + \alpha(U^{n+1} - 2U^n + \overline{U}^{n-1})$

Leap-Frog smoothed with the Robert-Asselin time filter; α~1%

b) $\dfrac{U^{n+1} - U^n}{\Delta t} = F(U^n)$ Euler (forward, good for diffusive terms,

unstable for hyperbolic equations)

c) $\dfrac{U^{n+1} - U^n}{\Delta t} = F(\dfrac{U^n + U^{n+1}}{2})$ Crank-Nicholson or centered

implicit

c') $\dfrac{U^{n+1} - U^n}{\Delta t} = F(\dfrac{\beta U^n + (1-\beta)U^{n+1}}{2}); \beta < 0.5$ Implicit, slightly diffusive

d) $\dfrac{U^{n+1} - U^n}{\Delta t} = F(U^{n+1})$ Fully implicit or backward, very diffusive

e) $\dfrac{U^* - U^n}{\Delta t} = F(U^n)$; $\dfrac{U^{n+1} - U^n}{\Delta t} = F(U^*)$ Euler-backward or Matsuno: good

for damping high frequency waves

f) $\dfrac{U^* - U^n}{\Delta t} = F(U^n)$; $\dfrac{U^{n+1} - U^n}{\Delta t} = F(\dfrac{U^n + U^*}{2})$ Another predictor-

corrector scheme (Heun)

g) $\dfrac{U^{n+1} - U^n}{\Delta t} = F(\dfrac{3}{2}U^n - \dfrac{1}{2}U^{n-1})$ Adams-Bashford (second order in

time).

h)

$$\frac{U^{n+1/2*} - U^n}{\Delta t / 2} = F(U^n) \, ;$$

$$\frac{U^{n+1/2**} - U^n}{\Delta t / 2} = F(U^{n+1/2*}) \, ;$$

$$\frac{U^{n+1*} - U^n}{\Delta t} = F(U^{n+1/2**})$$

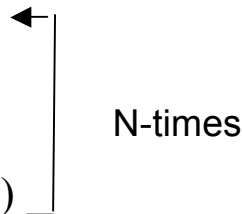$$\frac{U^{n+1} - U^n}{\Delta t} = \frac{1}{6}[F(U^n) + 2F(U^{n+1/2*}) + 2F(U^{n+1/2**}) + F(U^{n+1*})]$$

Runge-Kutta (fourth order), widely used for higher time accuracy (but expensive: 4 time derivatives, need to save many fields)

$$a = 0; \, b = 1/\Delta t$$

$$U* \leftarrow (aU* + F(U^n))/b$$

i) $U^n \leftarrow U^n + U*$

$$a \leftarrow a - 1/(N\Delta t); \, b \leftarrow b - 1/(N\Delta t)$$

N-times

Lorenz' N-cycle, N=multiple of 4; Nth order
Cheaper than Runge-Kutta but doesn't work so well…

j) $$\frac{U^{n+1} - U^{n-1}}{2\Delta t} = F_1(U^n) + F_2\left(\frac{U^{n+1} + U^{n-1}}{2}\right)$$

Semi-implicit

k) $$\frac{U^* - U^n}{\Delta t} = F_1(U^n); \, \frac{U^{n+1} - U^*}{\Delta t} = F_2(U*)$$

Fractional steps

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Note: Easy tests for hyperbolic and parabolic equations

1) It is easy to check the properties of these time schemes when applied to hyperbolic equations by testing them with a simple harmonic equation:

$$\frac{\partial U}{\partial t} = -i\nu U$$

with solution $U(t) = U(0)e^{-i\nu t}$.

After one time step, the exact solution is
$$U((n+1)\Delta t) = U(n\Delta t)e^{-i\nu\Delta t}$$

which indicates that the exact magnification factor is $e^{-i\nu\Delta t}$.

Example: forward scheme: $\dfrac{U^{n+1} - U^n}{\Delta t} = -i\nu U^n, \quad U^{n+1} = \rho U^n$

Therefore $\rho = 1 - i\nu\Delta t$ absolutely unstable for wave equations!

When we test a time scheme, we allow for space truncation errors that determine the value of the computational frequency ν , depending on the specific space discretization. For example, if we were using second order centered differences in space, and assume $U_j = A(t)e^{ik\Delta xj}$

$$c\frac{U_{j+1} - U_{j-1}}{2\Delta x} = A(t)e^{ik\Delta x j}\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} = U_j i \frac{\sin k\Delta x}{\Delta x}c$$

Comparing this with the true derivative

$c\frac{\partial U}{\partial x} = iUk\Delta x c$ we see that the computational speed for

second order differences is

$v = \frac{\sin k\Delta x}{\Delta x}c$, for a spectral scheme, $v = kc$.

For the fully implicit time scheme d),

$$\frac{U^{n+1} - U^n}{\Delta t} = F(U^{n+1}) = -ivU^{n+1}$$

the amplification factor is $\dfrac{1}{1 + iv\Delta t} = \dfrac{1 - iv\Delta t}{1 + (v\Delta t)^2}$.

Since the exact amplification factor has an amplitude equal to one, this shows that the implicit scheme is dissipative; similarly, comparing the imaginary components of the exact and approximate amplification factors, it is clear that the implicit solution is slowed down by a factor of about

$\dfrac{1}{1 + (v\Delta t)^2}$.

2) Equations with damping terms (such as the parabolic equation) can also be simply represented by the equation:

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$$\frac{\partial U}{\partial t} = -\mu U$$

Here μ can be considered as the computational rate of damping. For example, for the diffusion equation, using centered differences in space, $\mu = \frac{4\sigma}{(\Delta x)^2}\sin^2\frac{k\Delta x}{2}$.

Exercise 11: show that the leap-frog scheme is unstable for a damping term.

Exercise 12: write a numerically stable scheme for the equation with both wave-like and damping terms $\frac{\partial U}{\partial t} = -(i\nu + \mu)U$ using a 3-time level scheme.

Exercise 13: Show that for a wave equation the forward time scheme with centered differences in space is absolutely unstable. Note that this scheme shows that the "extrapolation" rule is a necessary but not a sufficient condition for stability of wave equations.

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3.2.5 Semi-implicit schemes

Consider the shallow water equations that we discussed in Section 2.4.1:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial \phi}{\partial y} - fu$$

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = -\Phi(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - (\phi - \Phi)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})$$

As indicated in that section, the phase speed of IGW is given by

$$c_{IGW} = \frac{v_{IGW}}{k} = U \pm \sqrt{\frac{f^2}{k^2} + \Phi} \approx U \pm 300m/\sec$$, and the terms that

give rise to the fast gravity waves are underlined. This

means that the Courant number $\mu = \dfrac{c_{IGW}\Delta t}{\Delta x}$ is dominated by

the speed of external gravity waves (equivalent to the Lamb waves, horizontal sound waves), and an explicit scheme would therefore require a time step an order of magnitude smaller than that required for advection. For this reason, Robert (1969) introduced the use of semi-implicit schemes to slow down the gravity waves. We write such a scheme using the compact finite difference notation for differences and averages:

$$\delta_x f = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}$$

$$\bar{f}^x = (f_{i+1/2} + f_{i-1/2})/2$$

and similarly for differences in y or t. With this notation, assuming uniform resolution,

$$\delta_{2x}f = \delta_x \bar{f}^x = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

$$\bar{f}^{2x} = (f_{i+1} + f_{i-1})/2$$

Using this compact finite difference notation we can write the Leap-Frog semi-implicit SWE as

$$\delta_{2t}u + u\delta_{2x}u + v\delta_{2y}u = -\delta_{2x}\bar{\phi}^{2t} + fv$$

$$\delta_{2t}v + u\delta_{2x}v + v\delta_{2y}v = -\delta_{2y}\bar{\phi}^{2t} - fu$$

$$\delta_{2t}\phi + u\delta_{2x}\phi + v\delta_{2y}\phi = -\Phi\overline{(\delta_{2x}u + \delta_{2y}v)}^{2t} - (\phi - \Phi)(\delta_{2x}u + \delta_{2y}v)$$

Everything that does not have a time average involves only terms evaluated explicitly at the n-th time step. We can rewrite the FDEs as

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = -\delta_{2x}(\phi^{n+1} + \phi^{n-1})/2 + R_u$$

$$\frac{v^{n+1} - v^{n-1}}{2\Delta t} = -\delta_{2y}(\phi^{n+1} + \phi^{n-1})/2 + R_v$$

$$\frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t} = -\Phi[\delta_{2x}(u^{n+1} + u^{n-1})/2 + \delta_{2y}(v^{n+1} + v^{n-1})/2] + R_\phi$$

where the "R" terms are the "rest" of the terms evaluated at the center time $n\Delta t$. For example,

$$R_u = fv - u\delta_{2x}u - v\delta_{2y}u$$, and similarly for R_v and R_ϕ.

From these three equations we can eliminate u^{n+1}, v^{n+1}, and obtain an elliptic equation for ϕ^{n+1} :

$$[\delta_{2x}^2 + \delta_{2y}^2 - \frac{1}{\Phi\Delta t^2}]\phi^{n+1} = -[\delta_{2x}^2 + \delta_{2y}^2 + \frac{1}{\Phi\Delta t^2}]\phi^{n-1} + 2[\delta_{2x}R_u + \delta_{2y}R_v]$$

$$+\frac{1}{\Delta t}\left[\delta_{2x}u^{n-1} + \delta_{2y}v^{n-1}\right] - \frac{2}{\Phi\Delta t}R_\phi = F_{i,j}^n$$

Note that the rhs of this elliptic equation is evaluated at t=nΔt or (n-1)Δt, so it is known. Solving this elliptic equation

provides ϕ^{n+1} , and once it is known, it can be plugged and solved for (u^{n+1}, v^{n+1}) .

The elliptic operator in brackets in the lhs is a FD equivalent to $(\nabla^2 - \lambda^2)$,

$$[\delta_{2x} + \delta_{2y} - \frac{1}{\Phi\Delta t^2}]\phi = \frac{\phi_{i+2,j} + \phi_{i-2,j} + \phi_{i,j+2} + \phi_{i,j-2} - 4(1+\frac{1}{\mu^2})\phi_{i,j}}{4\Delta^2}$$

 where we have assumed for simplicity that $\Delta x = \Delta y = \Delta$,

and $\mu^2 = \dfrac{\Phi\Delta t^2}{\Delta^2}$ is the square of the Courant number for gravity waves.

Since $\mu^2 = \dfrac{\Phi\Delta t^2}{\Delta^2} \gg 1$, the semi-implicit scheme distorts the GW solution, slowing the GW down until they satisfy the von Neumann criterion. This is an acceptable distortion since we

are interested in the slower "weather-like" processes and since the slower modes satisfy the CFL (von Neumann) stability criterion, and they are written explicitly, they are not slowed down or distorted in a significant way.

In the same way that the terms giving rise to gravity waves can be written semi-implicitly, the terms giving rise to sound waves can also be written semi-implicitly (Robert, 1993). They are the 3-dimensional divergence in the continuity equation (sections 2.3.2, 2.3.3). This has allowed the use of non-hydrostatic models without the use of the anelastic approximation or the hydrostatic approximation.

André Robert (1993) created a model that can be considered the "ultimate" atmospheric model. It treats the terms generating sound waves (anelastic terms, i.e., 3-dimensional divergence), and the terms generating gravity waves (pressure gradient and horizontal divergence) semi-implicitly, and it uses a 3-dimensional semi-Lagrangian scheme for all advection terms. This model, denoted "Mesoscale Compressible Community" (MCC) is a "universal" model designed so that it can tackle accurately atmospheric problems from planetary scale through mesoscale, convective and smaller (Laprise et al, 1997).

There is another approach followed by major non-hydrostatic models (e.g., MM5 and ARPS): the use of fractional steps (see Table 3.1, scheme k), with the sound waves terms integrated with small time steps. In addition, the ARPS model uses a semi-implicit scheme for vertically propagating sound waves (Xue et al, 1995).

Exercise 14: Consider the diffusion equation $\dfrac{\partial u}{\partial t} = \sigma \dfrac{\partial^2 u}{\partial x^2}$ with initial conditions $u = x$ for $x \le 0.5$ and $u = 1 - x$ for $x \ge 0.5$. 1) Compute the first 2 time steps using an explicit scheme (forward in time, centered in space) with 5 points between $x = 0 \text{ and } x = 1$, and a time step such that $r = \dfrac{\sigma \Delta t}{(\Delta x)^2}$ is equal to $r = 0.1, 0.5, 1.0$. Repeat using Crank-Nicholson's scheme.