Review of Probability

Wilks, Chapter 2

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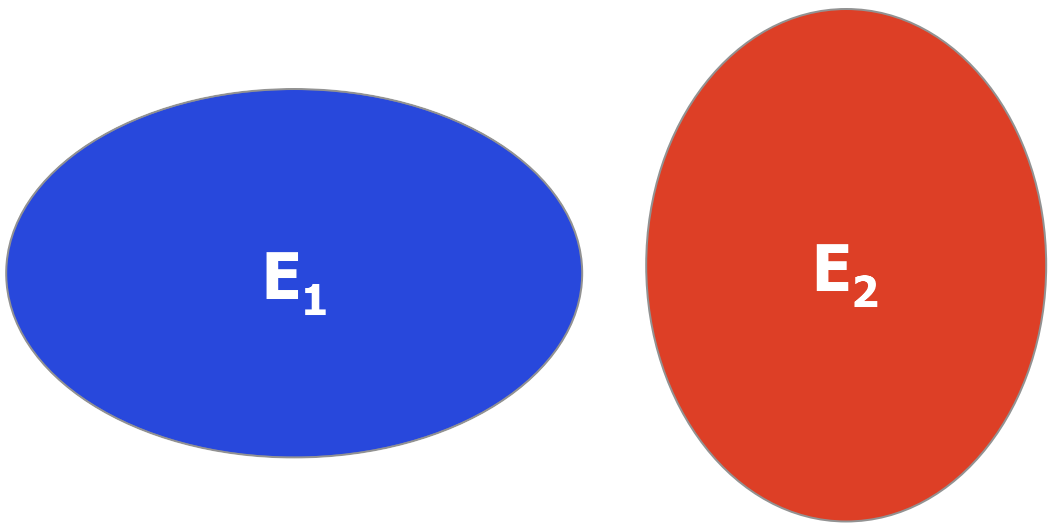
Definition

- **Event**: set, class, or group of possible uncertain outcomes
  - **A compound event** can be decomposed into two or more (sub) events
  - **An elementary event** cannot be decomposed
- **Sample space (event space), S**
  - The set of all possible elementary events
- **MECE (Mutually Exclusive & Collectively Exhaustive)**
  - **Mutually Exclusive**: no more than one of the events can occur
  - **Collectively Exhaustive**: at least one of the events will occur

➔ ***A set of MECE events completely fills a sample space***

Probability Axioms

- $P(A) \geq 0$
- $P(S) = 1$
- If $(E_1 \cap E_2)=0$, i.e., if E_1 and E_2 exclusive, then $P(E_1 \cup E_2)=P(E_1)+P(E_2)$

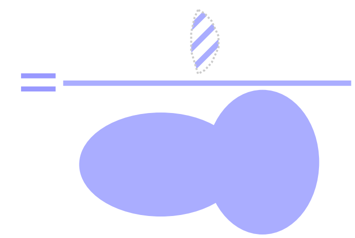


Probability

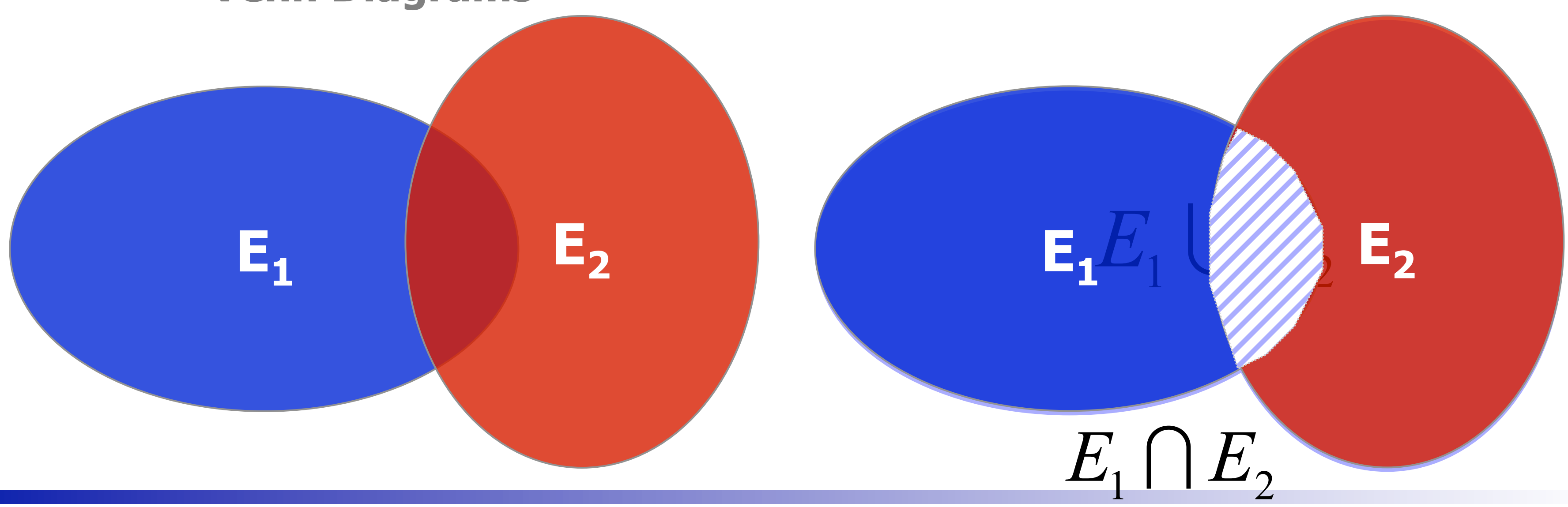
- **Probability ~ Frequency**
  - P(E) = $\lim_{n \to \infty} \frac{\# E = yes}{total\_n}$
- **If** $E_2 \subseteq E_1$**, then** $P(E_1) \geq P(E_2)$
- $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Recall Threat Score (TS)

$$= \frac{P(F = yes \cap Ob = yes)}{P(F = yes \cup Ob = yes)}$$



Venn Diagrams



Conditional Probability

- **Probability of E_1 given that E_2 has happened**

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cap E_2) = P(E_1|E_2)P(E_2)$$ **; Multiplicative Law**

- **Independent Event**
  - The occurrence or nonoccurrence of one does not affect the probability of the other

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

i.e. $$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = P(E_1)$$

Exercise

- **From the Penn State station data for Jan. 1980, compute the probability of precipitation, of T>32F, conditional probability of precipitation if T>32F, and conditional probability of precipitation tomorrow if it is raining today**

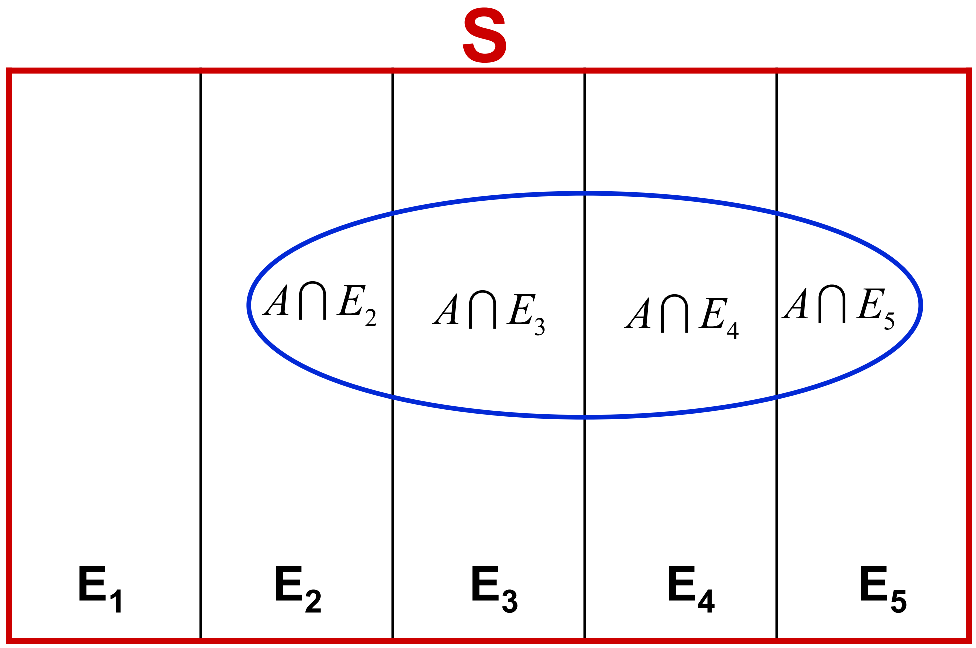
- **Prove graphically the DeMorgan Laws:**

$$P\left\{(A \cup B)^c\right\} = P\left\{A^c \cap B^c\right\} \; ; \; P\left\{(A \cap B)^c\right\} = P\left\{A^c \cup B^c\right\}$$

Total Probability

- **MECE events, $\{E_i\}$, $i=1, \ldots, I$**

$$P(A) = \sum_{i=1}^{I} P(A \cap E_i) = \sum_{i=1}^{I} P(A|E_i)P(E_i)$$



Bayes' Theorem

- **Bayes' theorem is used to "invert" conditional probabilities**
  - If $P(E_1|E_2)$ is known, Bayes' Theorem may be used to compute $P(E_2|E_1)$.

Multiplicative law

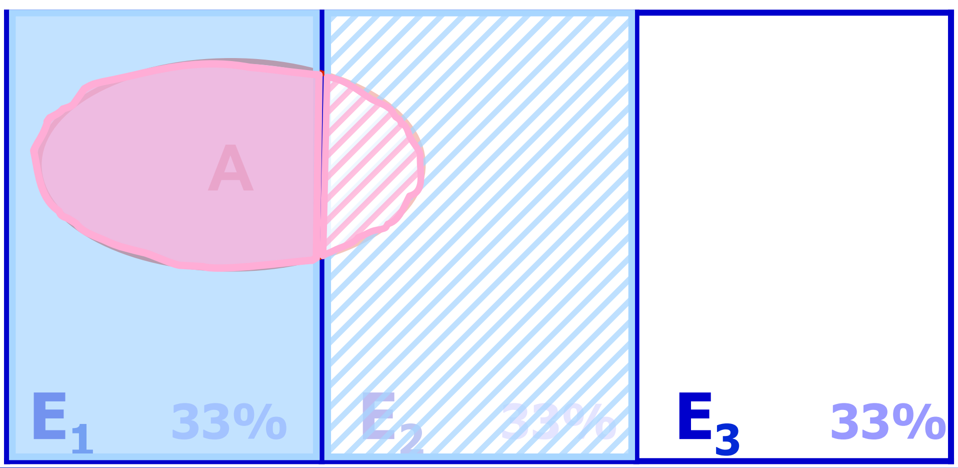
$$P(E_i|A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{I} P(A|E_i)P(E_j)}$$

Law of total probability

  - Combines prior information with new information

Example of Bayesian Reasoning

- **Relationship between precipitation over SE US and El Nino**
  - Precipitation Events: E_1(above), E_2(normal), E_3(below) are **MECE**
  - El Nino Event: A
  - ***Prior information*** (from past statistics)
    - ✓ **$P(E_1)=P(E_2)=P(E_3)=33\%$**
    - ✓ **$P(A|E_1)=40\%$; $P(A|E_2)=20\%$; $P(A|E_3)=0\%$**



$$P(A|E_1)=\frac{P(A\cap E_1)}{P(E_1)} = 0.40$$

$$P(A|E_2)=\frac{P(A\cap E_2)}{P(E_2)} = 0.20$$

Example of Bayesian Reasoning

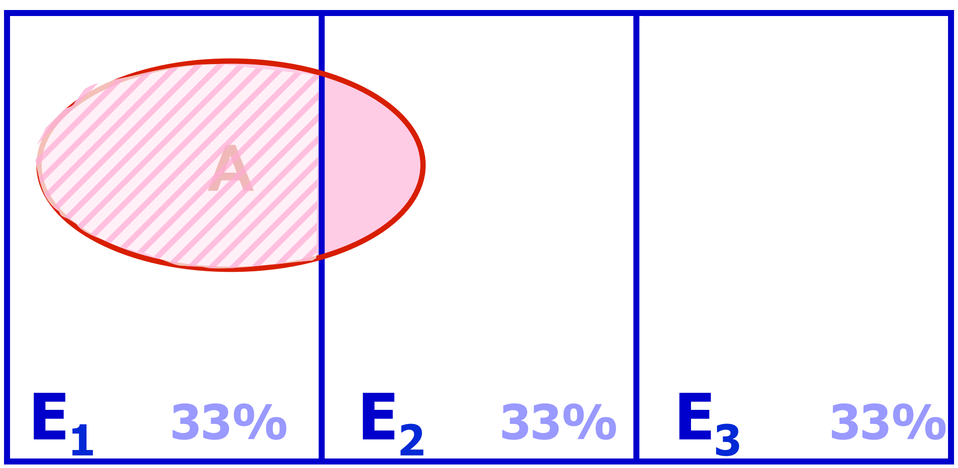
- **Total probability of A**

$$P(A) = \sum_{i=1}^{3} P(A \mid E_i) P(E_i)$$

$$= P(A \mid E_1)P(E_1) + P(A \mid E_2)P(E_2) + P(A \mid E_3)P(E_3)$$

$$= 0.4 * 0.33 + 0.2 * 0.33 + 0 * 0.33 = 0.20$$

- **NEW information: El Nino is happening! Probability of above normal precipitation?**



$$P(E_1 \mid A) = \frac{P(A \cap E_1)}{P(A)}$$

$$= \frac{0.4 * 0.33}{0.2} = 0.66 =$$

Probability Density Function when we have two observations, T_1 and T_2

With Gaussian errors, the best estimate of T is the weighted average of T_1 and T_2:

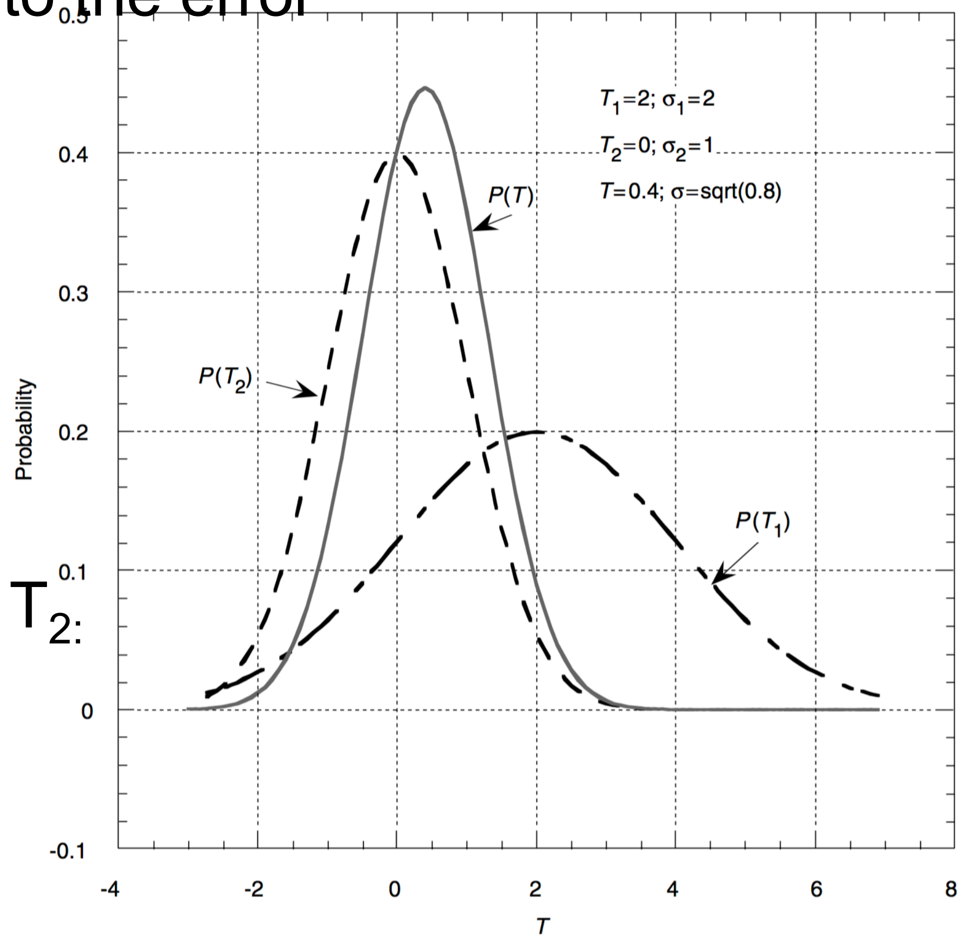
The weights are proportional to the error variance of the **other** obs.:

$$T = \frac{\sigma_2^2 T_1 + \sigma_1^2 T_2}{\sigma_1^2 + \sigma_2^2}$$

Precision=inverse of σ^2

Precision of T=
Precision of T_1 + Precision of T_2:

$$\frac{1}{\sigma_T^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$



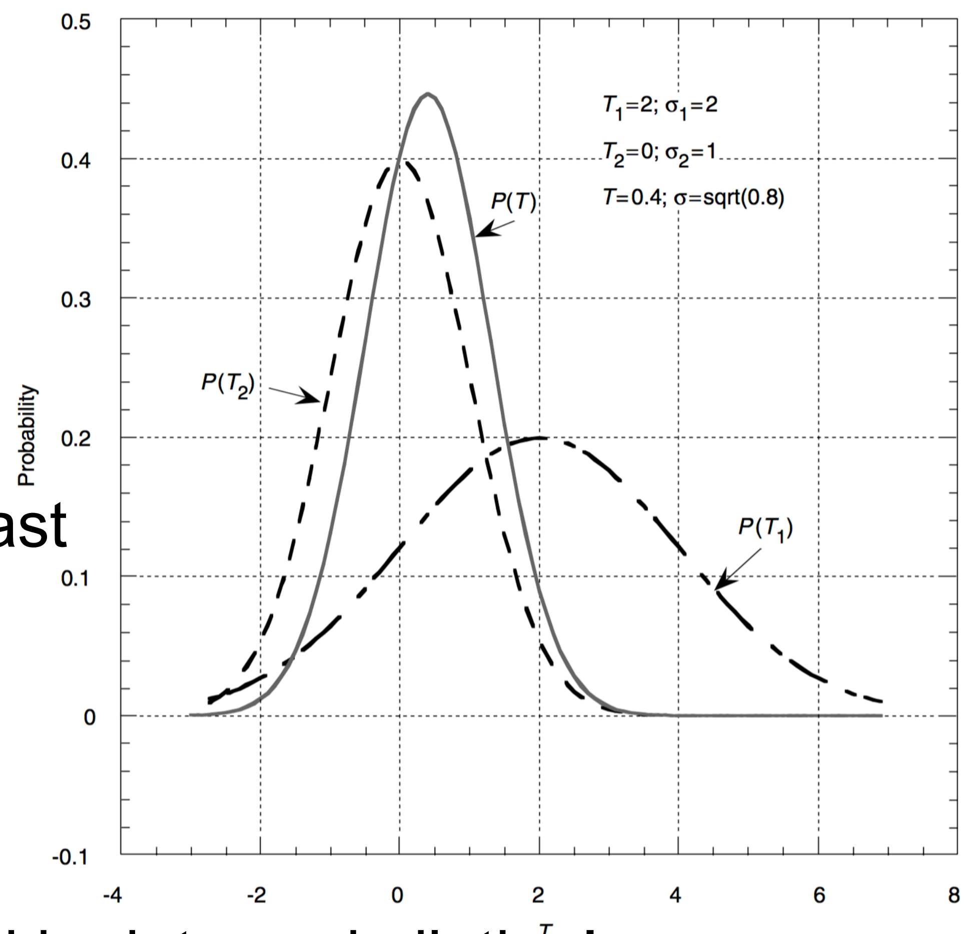
If we have a forecast T_f (prior info.) and then an observation T_o

The optimal analysis T_a is the best estimate of the truth (it has minimum errors):

$$T_a = \frac{\sigma_o^2 T_f + \sigma_f^2 T_o}{\sigma_f^2 + \sigma_o^2}$$

The analysis T_a is more accurate than both the forecast T_f and the observation T_o!

$$\frac{1}{\sigma_a^2} = \frac{1}{\sigma_f^2} + \frac{1}{\sigma_o^2}$$



These are the formulas used in data assimilation!

Now let's use a Bayesian approach for Data Assimilation

Bayes theorem $$P(T \mid T_o) = P(T \mid T_f)P(T_o \mid T) / P(T_o)$$

"The posterior probability of the true temperature T given the *prior* information T_f, and after receiving the new observation T_o, is given by the *prior* probability of T (based on the forecast T_f) multiplied by the *likelihood* of T given the observation T_o, normalized by the total probability of obtaining a measurement T_o."

The *likelihood* of T given the observation T_o is the same as the probability of observing T_o given a true temperature T (Edwards, 1984). This formula can be briefly read as:

"***posterior = prior . likelihood /normalization***"

The Bayesian approach for Data Assim is very general (not just Gaussians)

If we assume Gaussianity, the Bayes theorem leads to the Variational approach:

$$P(T \mid T_o) = P(T \mid T_f) P(T_o \mid T) / P(T_o) =$$

$$\frac{e^{-(T-T_f)^2/2\sigma_f^2}}{\sqrt{2\pi\sigma_f}} \frac{e^{-(T_o-T)^2/2\sigma_o^2}}{\sqrt{2\pi\sigma_o}} \frac{\sqrt{2\pi\sigma_o}}{e^{-(T_o-T_{cli})^2/2\sigma_o^2}} =$$

$$const * e^{-[(T-T_f)^2/2\sigma_f^2 + (T_o-T)^2/2\sigma_o^2]}$$

Since we want to maximize the probability of *T*, and T_{cli}, the climatological temperature probability distribution does not depend on *T*, the maximization can be written as the minimization of the exponent:

The Bayesian approach for Data Assim is very general (not just Gaussians)

From Bayes theorem, we minimize -the exponent, a cost function *J* that measures the squared distance between the optimal temperature *T* that we are seeking and the prior forecast, and with the new observation, both normalized by their error variances:

$$J = (T - T_f)^2 / 2\sigma_f^2 + (T_o - T)^2 / 2\sigma_o^2$$

Although the variational formulation looks very different from the classical formulation shown before, for Gaussian errors, both give the same solution. However, the Bayesian approach can be used with any probability distributions and allows the implementation of efficient "particle filters" (e.g., Penny and Miyoshi, 2016, Poterjoy, 2016).