Methods for finding coupled patterns in climate data

References: Bretherton, C., C. Smith and J.M. Wallace, 1992, J. of Climate, Danforth, Kalnay and Miyoshi (2007), MWR March. vandenDool (2007)

1) The EOF approach (aka Principal Component Analysis or PCA) is ideal for estimating dominant patterns in a single field (e.g., 500hPa) versus time:

$X(s,t)$ gridded maps anomalies, with $s = 1,\ldots,N_X, \quad t = 1,\ldots,T$

If we have two fields $X(s,t)$ and $Y(s,t)$ (e.g., SLP and SST), we can still use PCA by constructing the EOFs of $Z(s,t) = X(s,t) \mid Y(s,t)$, i.e., the vector that concatenates the two fields. Then the covariance of Z, or combined covariance has a size $(N_X + N_Y) \times (N_X + N_Y)$. As indicated before, it may be preferable to use a correlation matrix to avoid the problem of different units. This approach will give EOFs that do not necessarily represent the best coupled patterns.

2) Another approach is to generate correlation maps: take a time series (PC) from the EOFs of the first set of fields (or any other time series) and correlate it with the second field.

3) Canonical correlation analysis (CCA): it is designed to identify the linear combination of variables in one field most strongly correlated with linear combinations of variables of the second field. (See Barnett and Preisendorfer, MWR 1987).

4) Singular Value Decomposition. The cross-covariance matrix between $X(s,t)$ and $Y(s,t)$ is

$C_{XY} = < X(t)Y^T(t) >$ of elements: $c_{ij} = \dfrac{1}{T-1}\sum_{t=1}^{T} X(i,t)Y(j,t)$

It has size $(N_X \times N_Y)$ and is a non-square matrix if $N_X \neq N_Y$.

The properties of a non-square are made apparent by Singular Value Decomposition (SVD, Golub and van Loan, 1996), an extension of the *diagonalization* of a *square* matrix $\mathbf{M}$ using the matrix of the *eigenvectors* $\mathbf{Q}$: $\mathbf{Q}^T\mathbf{MQ} = diag(\lambda_1,...,\lambda_m)$

SVD: Given a matrix C of size $(N_X \times N_Y)$, there exist two square orthogonal matrices $U = [u_1 | u_2 | ... | u_{N_X}]$ and $V = [v_1 | v_2 | ... | v_{N_Y}]$ such that

$U^T CV = diag(\sigma_1,...,\sigma_p)$, (denoted SVD decomposition of C)

is a diagonal matrix of the positive singular values σ_i. Here p is the smaller of N_X and N_Y. The orthonormal vectors $u_1,...,u_p$ are the left singular vectors of C, and $v_1,...,v_p$ the right singular vectors of C, and correspond to the X and Y fields respectively.

The left and right singular vector have the properties that

$Cv_i = \sigma_i u_i$ and $C^T u_i = \sigma_i v_i$.

From these properties we can show that

$C^T Cv_i = \sigma_i^2 v_i$ and $CC^T u_i = \sigma_i^2 u_i$, so that u_i and v_i are the orthonormal eigenvectors of CC^T and $C^T C$ respectively, with the same eigenvalues σ_i^2. They are also the "coupled EOFs" of the two fields:

$$\tilde{X}(s,t) = \sum_{k=1}^{p} a_k(t)\, u_k(s)$$

$$\tilde{Y}(s,t) = \sum_{k=1}^{p} b_k(t)\, v_k(s)$$

where the time-dependent coefficients or "coupled PCs" can be computed by projection of the original fields on the "coupled EOFs":

$$a_k(t) = \sum_{s=1}^{N_X} X(s,t)\, u_k(s) \quad \text{and} \quad b_k(t) = \sum_{s=1}^{N_Y} Y(s,t)\, v_k(s)$$

The covariance between these time series is given by the singular values: $\langle a_k(t), b_k(t) \rangle = \sigma_k$, indicating the relative strength of the different coupled modes.

If we use only the K leading singular values, then the "coupled explained variance", i.e., the cumulative squared covariance fraction explained by these coupled modes, is given by

$$CSCF_K = \frac{\sum_{i=1}^{K} \sigma_i^2}{\sum_{i=1}^{p} \sigma_i^2}.$$

<u>Minimalist example:</u>

$$X = \begin{pmatrix} \sin t + \cos t \\ 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix}; \quad C = \overline{XY^T}^t = \begin{pmatrix} 1 & 0.5 \\ 0 & 0 \end{pmatrix}$$

(Note that $CC^T = \begin{pmatrix} 1.25 & 0 \\ 0 & 0 \end{pmatrix}; \quad C^T C = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.25 \end{pmatrix}$ are different symmetric matrices, and u, v are their corresponding eigenvectors).

The MATLAB instruction [U,S,V]=svd(C) yields

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad S = \begin{pmatrix} \sqrt{1.25} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1.180 & 0 \\ 0 & 0 \end{pmatrix}; \quad V = \begin{pmatrix} .8944 & -.4472 \\ .4472 & .8944 \end{pmatrix}$$

which satisfy $Cv_i = \sigma_i u_i$, $C^T u_i = \sigma_i v_i$, $C^T C v_i = \sigma_i^2 v_i$, and $CC^T u_i = \sigma_i^2 u_i$.

Only the first singular value $\sigma_1 = \sqrt{1.25}$ is different from zero, and yields the first mode that dominates the covariance between *X* and *Y.* The spatial patterns that covary are respectively the first left and right singular vectors,

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad v_1 = \begin{pmatrix} .8944 \\ .4472 \end{pmatrix}.$$

The "coupled PCs" or expansion coefficients are

$$a_1(t) = \sum_{s=1}^{2} X(s,t)\, u_1(s) = \sin(t) + \cos(t)$$

and $b_1(t) = \sum_{s=1}^{2} Y(s,t)\, v_1(s) = \dfrac{4\sin(t) + \cos(t)}{\sqrt{5}} = 1.7889\sin(t) + .4472\cos(t)$

and their covariance is $\langle a_1(t), b_1(t) \rangle = \sigma_1 = \dfrac{2.5}{\sqrt{5}} = 1.1180$.

The covarying component of the original fields can be represented by

$$\tilde{X}(s,t) = a_1(t) u_1(s) = \begin{pmatrix} \sin(t) + \cos(t) \\ 0 \end{pmatrix}$$

and

$$\tilde{Y}(s,t) = b_1(t) v_1(s) = \begin{pmatrix} 1.6\sin(t) + 0.4\cos(t) \\ 0.8\sin(t) + 0.2\cos(t) \end{pmatrix}$$

Recall that the original fields were

$$X = \begin{pmatrix} \sin t + \cos t \\ 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix}$$

A more realistic example is in Danforth et al (2007) who found the systematic forecast errors that are dependent on the state of the model by doing SVD of the covariance of 6 hour forecast errors and model state anomalies. The figure shows the coupled forecast state anomalies (contours) and corresponding forecast errors, with their corresponding time correlations in color. It shows that the errors indicate for each of the features whether they are too strong, too weak, or shifted East or West. The numbers correspond to the leading three coupled EOFs.

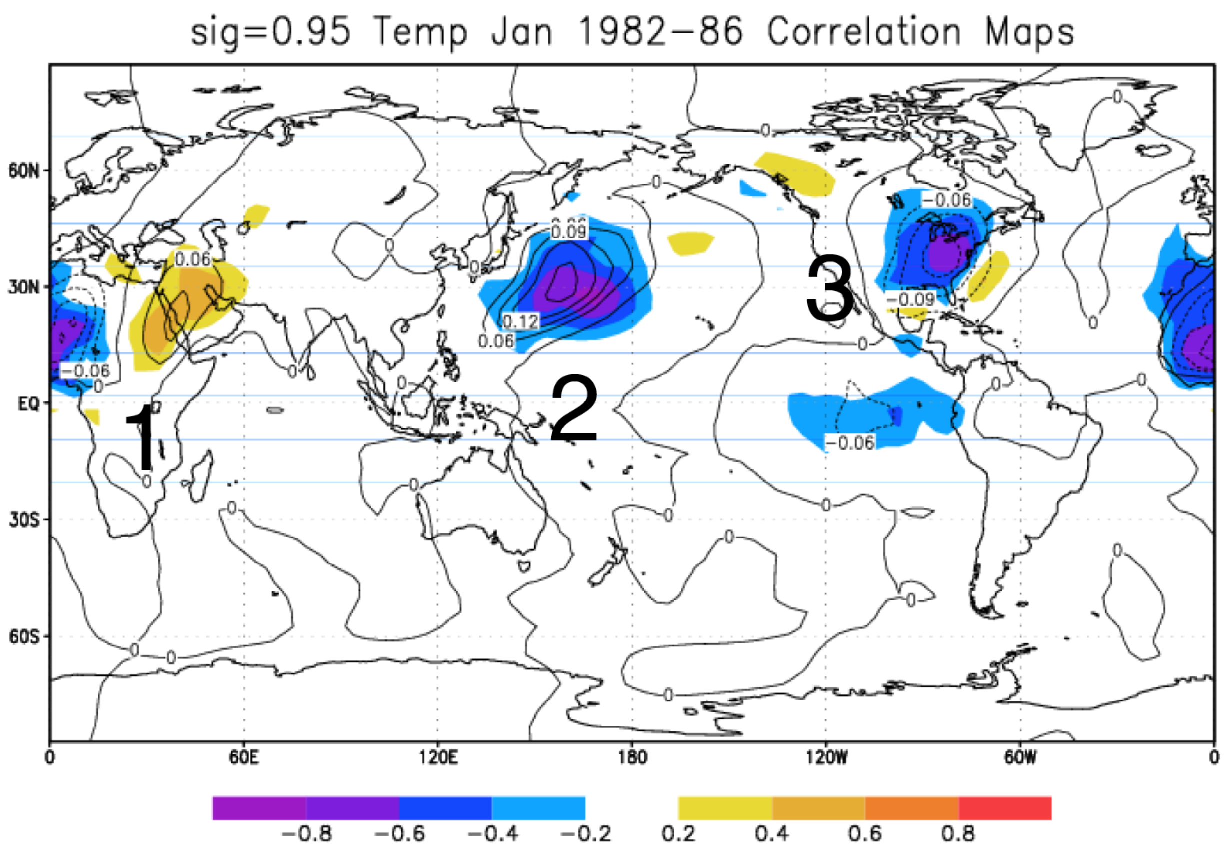

Figure 1: Three leading coupled SVD's of the covariance of 6 hr forecast errors and corresponding model state anomaly for T at sigma=0.95. Contours: state anomaly, colors: heterogeneous correlation with forecast errors. Note that over land, the corrections suggest the anomalous temperatures are too strong, and over ocean too weak and too far to the west.

Use of coupled SVDs for forecasting:

Assume that we have computed the SVD expansion using a training period and we want to estimate the field of *X* explained by it covarying component, given a new field of *Y* valid at time T. We would compute the (heterogeneous) correlation between the *X* field and the *b(t)* for the dependent sample, which is given by

$\rho\left[X(s,t), b_k(t)\right] = \left(\dfrac{\sigma_k}{\left\langle b_k^2(t) \right\rangle^{1/2} \left\langle X^2(s,t) \right\rangle^{1/2}} \right) u_k(s)$, and is proportional to the singular value and to the spatial pattern of the *k* singular vector corresponding to *X*.

In order to predict the field *X(s,T)* given a new *Y(s,T)*, we compute the new expansion coefficient of *Y(S,T)*

$$b_k(T) = \sum_{s=1}^{N_Y} Y(s,T)\, v_k(s)$$

Then the estimated field of *X(s,t)* corresponding to the first *K* singular values will be given by

$$X_{pred}(s,T) = \sum_{k=1}^{K} \rho\left[X(s,t), b_k(t)\right] \sqrt{\frac{\left\langle X^2(s,t) \right\rangle}{\left\langle b_k^2(t) \right\rangle}}\, b_k(T) = \sum_{k=1}^{K} \frac{\sigma_k}{\left\langle b_k^2(t) \right\rangle} u_k(s) b_k(T)$$

where t refers to the training period and T to the forecast time.

The following figure is an example of the correction that the SVDs would suggest (it takes place only in areas in which the forecast anomaly projects strongly on the coupled forecast SVDs, elsewhere there is no correction).

Figure 2: Top right: contours: coupled anomalous state singular vector and colors: corresponding correlation map with the error (from the training period). Bottom left: New forecast (contours) and actual forecast errors (colors). Bottom right: in the area where the forecast field projects on the forecast SVD, it is possible to substantially correct the forecast errors.